14 Pricing

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14.1 Quantities as Strategies: Cournot Equilibrium Revisited

Chapter 14 is about how firms with market power set prices. Section 14.1 extends the Cournot Game of Section 3.5 in which two firms choose the quantities they sell, while Section 14.2 extends the Bertrand model in which they choose prices to the case where capacity is limited. Section 14.3 goes back to the origins of product differentiation, and develops two Hotelling location models. Section 14.4 shows how to do comparative statics in games, using the differentiated Bertrand model as an example and supermodularity and the implicit function theorem as tools. Section 14.5 looks at another sort of differentiation: choice of "vertical" quality, from good to bad, by monopoly or duopoly. Section 14.6 concludes this book with the problem facing a firm selling a durable good because buyers foresee that it will be tempted to reduce the price over time to price-discriminate among them. At that point, perhaps you'll wonder how much this book will cost next year!

Cournot Behavior with General Cost and Demand Functions

In the next few sections, sellers compete against each other while moving simultaneously. We will start by generalizing the Cournot Game of Section 3.5 from linear demand and zero costs to a wider class of functions. The two players are firms Apex and Brydox, and their strategies are their choices of the quantities q_a and q_b. The payoffs are based on the total cost functions, $c(q_a)$ and $c(q_b)$, and the demand function, $p(q)$, where $q = q_a + q_b$. This specification says that only the sum of the outputs affects the price. The implication is that the firms produce an identical product, because whether it is Apex or Brydox that produces an extra unit, the effect on the price is the same.

Let us take the point of view of Apex. In the Cournot-Nash analysis, Apex chooses its output of q_a for a given level of q_b as if its choice did not affect q_b. From its point of view, q_a is a function of q_b, but q_b is exogenous. Apex sees the effect of its output on price as

$$\frac{\partial p}{\partial q_a} = \left(\frac{dp}{dq}\right)\left(\frac{\partial q}{\partial q_a}\right) = \frac{dp}{dq}. \tag{1}$$

Apex's payoff function is

$$\pi_a = p(q)q_a - c(q_a). \tag{2}$$

To find Apex's reaction function, we differentiate with respect to its strategy to obtain

$$\frac{d\pi_a}{dq_a} = p + \left(\frac{dp}{dq}\right) q_a - \frac{dc}{dq_a} = 0, \tag{3}$$

which implies

$$q_a = \frac{\frac{dc}{dq_a} - p}{\frac{dp}{dq}}, \tag{4}$$

or, simplifying the notation,

$$q_a = \frac{c' - p}{p'}. \tag{5}$$

If particular functional forms for $p(q)$ and $c(q_a)$ are available, equation (5) can be solved to find q_a as a function of q_b. More generally, to find the change in Apex's best response for an exogenous change in Brydox's output, differentiate (5) with respect to q_b, remembering that q_b exerts not only a direct effect on $p(q_a + q_b)$, but possibly an indirect effect via q_a.

$$\frac{dq_a}{dq_b} = \frac{(p - c')\left(p'' + p''\left(\frac{dq_a}{dq_b}\right)\right)}{p'^2} + \frac{c''\left(\frac{dq_a}{dq_b}\right) - p' - p'\left(\frac{dq_a}{dq_b}\right)}{p'}. \tag{6}$$

Equation (6) can be solved for $\frac{dq_a}{dq_b}$ to obtain the slope of the reaction function,

$$\frac{dq_a}{dq_b} = \frac{(p - c')p'' - p'^2}{2p'^2 - c''p' - (p - c')p''} \tag{7}$$

If both costs and demand are linear, as in section 3.5, then $c'' = 0$ and $p'' = 0$, so equation (7) becomes

$$\frac{dq_a}{dq_b} = -\frac{p'^2}{2p'^2} = -\frac{1}{2}. \tag{8}$$

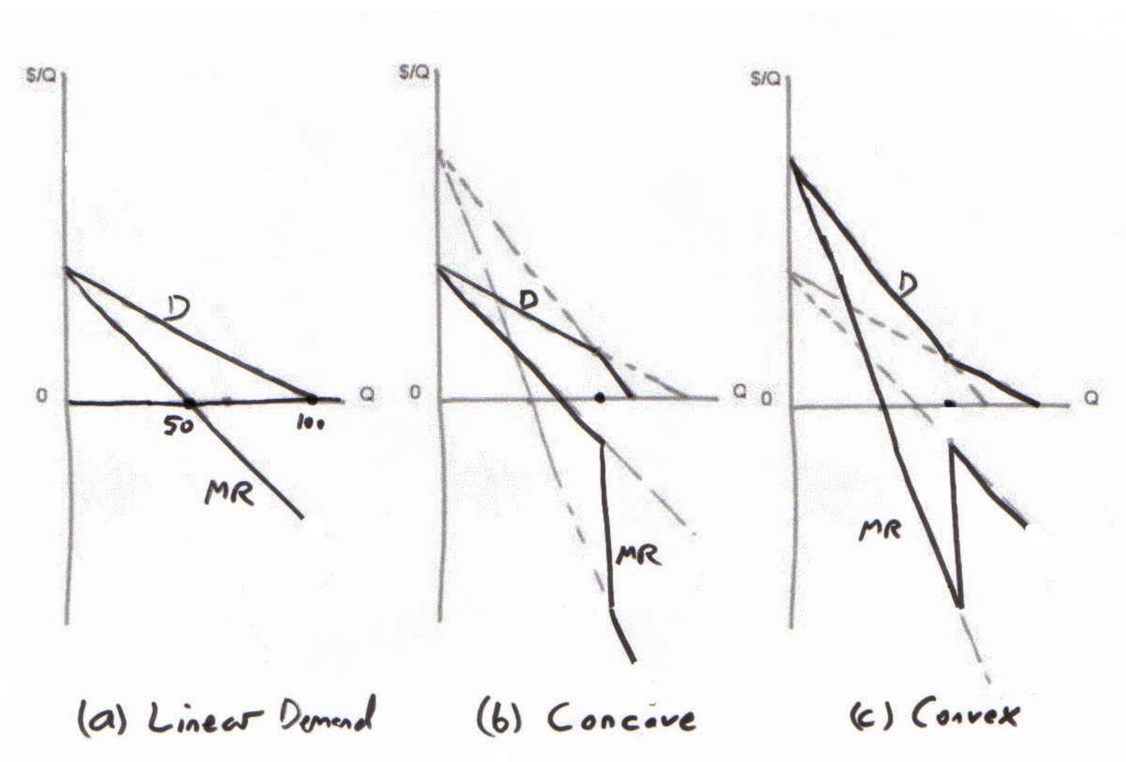

Figure 1: Different Demand Curves

The general model faces two problems that did not arise in the linear model: nonuniqueness and nonexistence. If demand is concave and costs are convex, which implies that $p'' < 0$ and $c'' > 0$, then all is well as far as existence goes. Since price is greater than marginal cost ($p > c'$), equation (7) tells us that the reaction functions are downward sloping, because $2p'^2 - c''p' - (p - c')p''$ is positive and both $(p - c')p''$ and $-p'^2$ are negative. If the reaction curves are downward sloping, they cross and an equilibrium exists, as was shown in Figure 1a for the linear case represented by equation (8). We usually do assume that costs are at least weakly convex, since that is the result of diminishing or constant returns, but there is no reason to believe that demand is either concave as in Figure 1b or convex, as in Figure 1c. If the demand curves are not linear, the contorted reaction functions of equation (7) might give rise to multiple Cournot equilibria as in Figure 2.

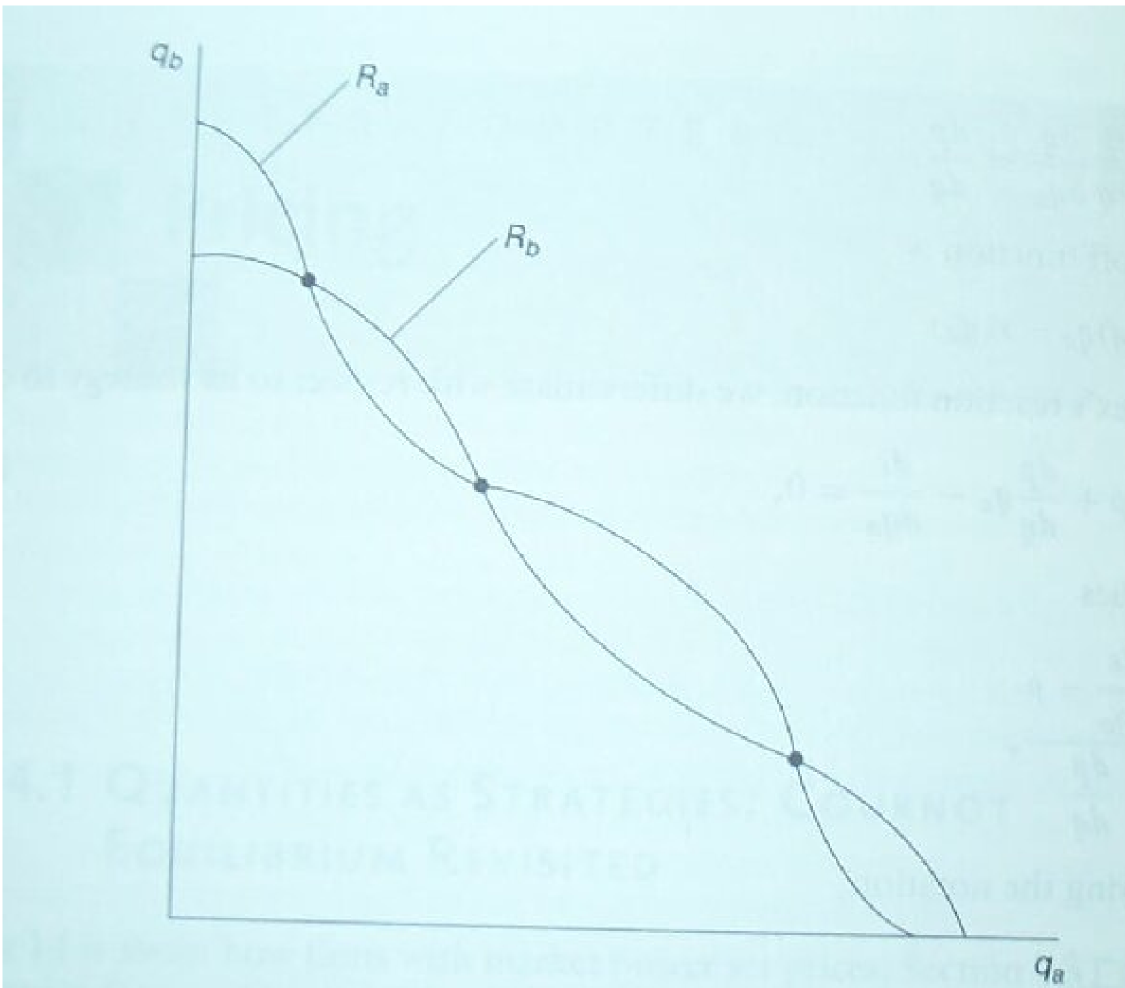

Figure 2: Multiple Cournot-Nash Equilibria

If demand is convex or costs are concave, so $p'' > 0$ or $c'' < 0$, the reaction functions can be upward sloping, in which case they might never cross and no equilibrium would exist. The problem can also be seen from Apex's payoff function, equation (2). If $p(q)$ is convex, the payoff function might not be concave, in which case standard maximization techniques break down. The problems of the general Cournot model teach a lesson to modellers: sometimes simple assumptions such as linearity generate atypical results.

Many Oligopolists

Let us return to the simpler game in which production costs are zero and demand is linear. For concreteness, we will use the same specific inverse demand function as in Chapter 3,

$$p(q) = 120 - q. \tag{9}$$

Using (9), the payoff function, (2), becomes

$$\pi_a = 120q_a - q_a^2 - q_b q_a. \tag{10}$$

In section 3.5, firms picked outputs of 40 apiece given demand function (9). This generated a price of 40. With n firms instead of two, the demand function is

$$p\left(\sum_{i=1}^{n} q_i\right) = 120 - \sum_{i=1}^{n} q_i, \tag{11}$$

and firm j's payoff function is

$$\pi_j = 120q_j - q_j^2 - q_j \sum_{i \neq j} q_i. \tag{12}$$

Differentiating j's payoff function with respect to q_j yields

$$\frac{d\pi_j}{dq_j} = 120 - 2q_j - \sum_{i \neq j} q_i = 0. \tag{13}$$

The first step in finding the equilibrium is to guess that it is symmetric, so that $q_j = q_i, (i = 1, \ldots, n)$. This is an educated guess, since every player faces a first-order condition like (13). By symmetry, equation (13) becomes $120 - (n+1)q_j = 0$, so that

$$q_j = \frac{120}{n+1}. \tag{14}$$

Consider several different values for n. If $n = 1$, then $q_j = 60$, the monopoly optimum; and if $n = 2$ then $q_j = 40$, the Cournot output found in section 3.5. If $n = 5$, $q_j = 20$; and as n rises, individual output shrinks to zero. Moreover, the total output of $nq_j = \left(\frac{n}{n+1}\right) 120$ gradually approaches 120, the competitive output, and the market price falls to zero, the marginal cost of production. As the number of firms increases, profits fall.

14.2 Capacity Constraints: The Edgeworth Paradox

In the last section we assumed constant marginal costs (of zero), and we assumed constant marginal costs of 12 in Chapter 3 when we first discussed Cournot and Bertrand equilibrium. What if it were increasing, either gradually, or abruptly rising to infinity at a fixed capacity?

In the Cournot model, where firms compete in quantities, increasing marginal costs or a capacity constraint complicate the equations but do not change the model's features dramatically. Increasing marginal cost would reduce output as one might expect. If one firm had a capacity that was less than the ordinary Cournot output, that firm would produce only up to its capacity and the other firm would produce more than the ordinary Cournot output, since their outputs are strategic substitutes.

What happens in the Bertrand model, where firms compete in prices, is less straightforward. In Chapter 3's game, the demand curve was $p(q) = 120 - q$, which we also used in the previous section of this chapter, and the constant marginal cost of firms Apex and Brydox was $c = 12$. In equilibrium, $p_a = p_b = 12$ and $q_a = q_b = 54$. If Apex deviated to a higher price such as $p_a = 20$, its quantity would fall to zero, since all customers would prefer Brydox's low price.

What happens if we constrain each firm to sell no more than its capacity of $K_a = K_b = 70$? The industry capacity of 140 continues to exceed the demand of 108 at $p_a = p_b = 12$. If, however, Apex deviates to the higher price of $p_a = 20$, it can still get customers. All 108 customers would prefer to buy from Brydox, but Brydox could only serve 70 of them, and the rest would have to go unhappily to Apex.

To discover what deviation is most profitable for Apex when $p_a = p_b = 12$, however, we need to know what Apex's exact payoff would be from deviation. That means we need to know not only that 38 (108 minus 70) of the customers are turned away by Brydox, but which 38 customers. If they are the customers at the top of the demand curve, who are willing to pay prices near 100, Apex's optimal deviation will be much different than if they are ones towards the bottom, who are only willing to pay prices a little above 12.

Thus, in order to set up the payoff functions for the game, we need to specify a **rationing rule** to tell us which consumers are served at the low price and which must buy from the high-price firm. The rationing rule is unimportant to the payoff of the low-price firm, but crucial to the high-price firm.

One possible rule is

Intensity rationing (or **efficient rationing**, or **high-to-low rationing**). *The consumers able to buy from the firm with the lower price are those who value the product most.*

The inverse demand function from equation (9) is $p = 120 - q$, and under intensity rationing the K consumers with the strongest demand buy from the low-price firm. Suppose that Brydox is the low-price firm, charging (for illustration) a price of $p_b = 30$, so 90 consumers wish to buy from it though only K can do so, and Apex is charging some higher price p_a. The residual demand facing Apex is either 0 (if $p_a > 120 - K$) or

$$q_a = 120 - p_a - K. \tag{15}$$

That is the demand curve in Figure 3(a).

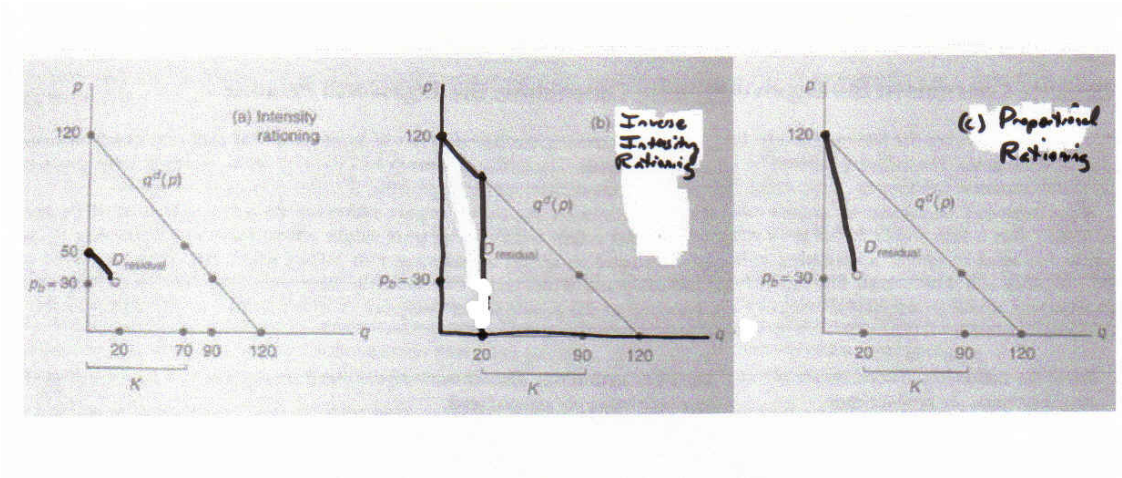

Figure 3: Rationing Rules when $p_b = 30$, $p_a > 30$, and $K = 70$

Under **intensity rationing**, if $K = 70$ the demand function for Apex (Brydox's is analogous) is

$$q_a = \begin{cases} Min\{120 - p_a, 70\} & \text{if } p_a < p_b & (a) \\ \frac{120-p_a}{2} & \text{if } p_a = p_b & (b) \\ Max\{(120 - p_a - 70), 0\} & \text{if } p_a > p_b, p_b < 50 & (c) \\ 0 & \text{if } p_a > p_b, p_b \geq 50 & (d) \end{cases} \quad (16)$$

Equation (16a) is true because if Apex has the lower price, all consumers will want to buy from Apex if they buy at all. All of the $(120 - p_a)$ customers who want to buy at that price will be satisfied if there are 70 or less; otherwise only 70. Equation (16b) simply says the two firms split the market equally if prices are equal. Equation (16c) is true because if Brydox's price is the lowest and is less than 50, Brydox will sell 70 units, and the residual demand curve facing Apex will be as in equation (15). If Brydox's price is the lowest but exceeds 50, then less than 70 customers will want to buy at all, so Brydox will satisfy all of them and zero will be left for Apex – which is equation (16d).

The appropriate rationing rule depends on what is being modelled. Intensity rationing is appropriate if buyers with more intense demand make greater efforts to obtain low prices. If the intense buyers are wealthy people who are unwilling to wait in line, the least intense buyers might end up at the low-price firm which is the case of the **inverse-intensity rationing** (or **low-to-high rationing**) in Figure 3b. An intermediate rule is proportional rationing, under which every type of consumer is equally likely to be able to buy at the low price.

Proportional rationing. *Each consumer has the same probability of being able to buy from the low-price firm.*

Under proportional rationing, if $K = 70$ and 90 consumers wanted to buy from Brydox, 2/9 $(= \frac{q(p_b)-K}{q(p_b)})$ of each type of consumer will be forced to buy from Apex (for example, 2/9 of the type willing to pay 120). The residual demand curve facing Apex, shown in Figure 3c and equation (17), intercepts the price axis at 120, but slopes down at a rate three times as fast as market demand because there are only 2/9 as many remaining consumers of each type.

$$q_a = (120 - p_a)\left(\frac{120 - p_b - K}{120 - p_b}\right) \tag{17}$$

We thus have three choices for rationing rules, with no clear way to know which to use. Let's use intensity rationing. That is the rule which makes deviation to high prices least attractive, since the low-price firm keeps the best customers for itself, so if we find that the normal Bertrand equilibrium breaks down there, we will know it would break down under the other rationing rules too.

The Bertrand Game with Capacity Constraints

Players
Firms Apex and Brydox

The Order of Play
Apex and Brydox simultaneously choose prices p_a and p_b from the set $[0, \infty)$.

Payoffs
Marginal cost is constant at $c = 12$. Demand is a function of the total quantity sold, $Q(p) = 120 - p$. The payoff function for Apex (Brydox's would be analogous) is, using equation (16) for q_a,

$$\pi_a = \begin{cases} (p_a - c) \cdot Min\{120 - p_a, 70\} & \text{if } p_a < p_b & (a) \\ (p_a - c)\left(\frac{120 - p_a}{2}\right) & \text{if } p_a = p_b & (b) \\ (p_a - c) \cdot Max\{(120 - p_a - 70), 0\} & \text{if } p_a > p_b, p_b < 50 & (c) \\ 0 & \text{if } p_a > p_b, p_b \geq 50 & (d) \end{cases} \tag{18}$$

The capacity constraint has a very important effect: $(p_a = 12, p_b = 12)$ is no longer a Nash equilibrium in prices, even though the industry capacity of 140 is well over the market demand of 108 when price equals marginal cost. Apex's profit would be zero in that strategy profile. If Apex increased its price to $p_a = 20$, Brydox would immediately sell $q_b = 70$, and to the most intense 70 of buyers. Apex would be left with all the buyers between $p_a = 20$ and $p_a = 12$ on the demand curve for sales of $q_a = 30$ and a payoff of 240 from equation (18c). So deviation by Apex is profitable. (Of course, $p_a = 20$ is not necessarily the most profitable deviation — but we do not need to check that; any profitable deviation is enough to refute the proposed equilibrium.)

Equilibrium prices must be lower than 120, because that price yields a zero payoff under any circumstance. There are three remaining possibilities (now that we have ruled out $p_a = p_b = 12$) for prices chosen in the open interval [12,120).

(i) Equal prices with $p_a = p_b > 12$ are not an equilibrium. Even if the price is close to 12, Apex would sell at most 54 units as its half of the market, which is less than its capacity of 70. Apex could profitably deviate to just below p_b and have a discontinuous jump in sales for an increase in profit, just as in the basic Bertrand game.

(ii) Unequal prices with one equal to 12 are not an equilibrium. Without loss of generality, suppose $p_a > p_b = 12$. Apex could not profitably deviate, but Brydox could deviate to $p_b = p_a - \epsilon$ and make positive instead of zero profit.

(iii) Finally, unequal prices of (p_a, p_b) with both greater than 12 are not an equilibrium. Without loss of generality, suppose $p_a > p_b > 12$. Apex's profits are shown in equation (18c). If $\pi_a = 0$, it can gain by deviating to $p_a = p_b - \epsilon$. If $\pi_a = (p_a - c)(50 - p_a)$, it can gain by deviating to $p_a = p_b - \epsilon$, because equation (18c) tells us that Apex's payoff will rise to either $\pi_a = (p_a - c)(70)$ (if $p_b \geq 50$) or $\pi_a = (p_a - c)(120 - p_a)$ (if $p_b < 50$).

Thus, no equilibrium exists in pure strategies under intensity rationing, and similar arguments rule out pure-strategy equilibria under other forms of rationing. This is known as the **Edgeworth paradox** after Edgeworth (1897, 1922).

Nowadays we know that the resolution to many a paradox is mixed strategies, and that is the case here too. A mixed-strategy equilibrium does exist, calculated using intensity rationing and linear demand by Levitan & Shubik (1972). Expected profits are positive, because the firms charge prices above marginal cost. In the symmetric equilibrium, the firms mix using distribution $F(p)$ with a support $[\underline{p}, \bar{p}]$, where $\underline{p} > c$ and $\bar{p}$ is the monopoly price for the residual demand curve (15), which happens to be $\bar{p} = 36$ in our example. The upper bound $\bar{p}$ is that monopoly price because $F(\bar{p}$ and the firm choosing that price certainly is the one with the highest price and so should maximize its profit using the residual demand curve. The payoff from playing the lower bound, $\underline{p}$, is $(\underline{p} - c)(70)$ from equation (18c), so since that payoff must equal the payoff of 336 (=(p-c)q= (36-12) (50-36)) from $\bar{p}$, we can conclude that $(\underline{p} = 16.8$. The mixing distribution $F(p)$ could then be found by setting $\pi(p) = 336 = F(p)(p - c)(50 - p) + (1 - F(p))(p - c)(70)$ and solving for $F(p)$.

If capacities are large enough— above the capacity of $K = Q(c) = 108$ in this example— the Edgeworth paradox disappears. The argument made above for why equal prices of c is not an equilibrium fails, because if Apex were to deviate to a positive price, Brydox would be fully capable of serving the entire market, leaving Apex with no consumers.

If capacities are small enough— less than $K = 36$ in our example— the Edgeworth paradox also disappears, but so does the Bertrand paradox. The equilibrium is in pure strategies, with each firm using its entire capacity, so $q_a = q_b = K$ and charging the same price. There is no point in a firm reducing its price, since it cannot sell any greater quantity. How about a deviation to increasing its price and reducing its quantity? Its best deviation is to the price which maximizes profit using the residual demand curve $(120 - K - p)$. This turns out to be $p^* = (120 - K + c)/2$, in which case

$$q^* = \frac{120 - K - c}{2}. \tag{19}$$

But if

$$K < \frac{120 - c}{3}, \tag{20}$$

then $q^* > K$ in equation (19) and is infeasible— the profit- maximizing price is from using all the capacity. The critical level from inequaity (20) is $K = 36$ in our example. For any lower capacities, firms simply dump their entire capacity onto the market and the price, $p_a = p_b = 120 - 2K$, exceeds marginal cost.

14.3 Location Models

In Chapter 3 we analyzed the Bertrand model with differentiated products using demand functions whose arguments were the prices of both firms. Such a model is suspect because it is not based on primitive assumptions. In particular, the demand functions might not be generated by maximizing any possible utility function. A demand curve with a constant elasticity less than one, for example, is impossible because as the price goes to zero, the amount spent on the commodity goes to infinity. Also, the demand curves were restricted to prices below a certain level, and it would be good to be able to justify that restriction.

Location models construct demand functions like those in Chapter 3 from primitive assumptions. In location models, a differentiated product's characteristics are points in a space. If cars differ only in their mileage, the space is a one-dimensional line. If acceleration is also important, the space is a two-dimensional plane. An easy way to think about this approach is to consider the location where a product is sold. The product "gasoline sold at the corner of Wilshire and Westwood," is different from "gasoline sold at the corner of Wilshire and Fourth." Depending on where consumers live, they have different preferences over the two, but, if prices diverge enough, they will be willing to switch from one gas station to the other.

Location models form a literature in themselves. We will look at the first two models analyzed in the classic article of Hotelling (1929), a model of price choice and a model of location choice. Figure 4 shows what is common to both. Two firms are located at points x_a and x_b along a line running from zero to one, with a constant density of consumers throughout. In the Hotelling Pricing Game, firms choose prices for given locations. In the Hotelling Location Game, prices are fixed and the firms choose the locations.

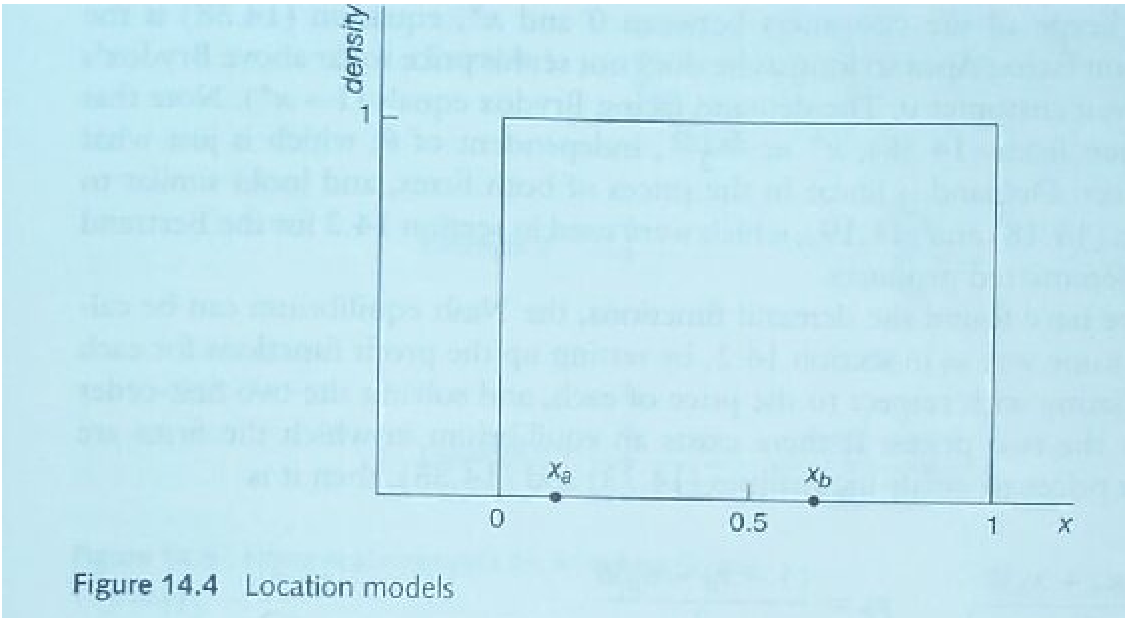

Figure 14.4 Location models

Figure 4: Location Models

The Hotelling Pricing Game

(Hotelling [1929])

Players

Sellers Apex and Brydox, located at x_a and x_b, where $x_a < x_b$, and a continuum of buyers indexed by location $x \in [0, 1]$.

The Order of Play

1 The sellers simultaneously choose prices p_a and p_b.
2 Each buyer chooses a seller.

Payoffs

Demand is uniformly distributed on the interval [0,1] with a density equal to one (think of each consumer as buying one unit). Production costs are zero. Each consumer always buys, so his problem is to minimize the sum of the price plus the linear transport cost, which is θ per unit distance travelled.

$$\pi_{buyer\ at\ x} = V - Min\{\theta|x_a - x| + p_a,\ \theta|x_b - x| + p_b\}. \tag{21}$$

$$\pi_a = \begin{cases} p_a(0) = 0 & \text{if } p_a - p_b > \theta(x_b - x_a) \quad \text{(a)} \\ & \text{(Brydox captures entire market)} \\ p_a(1) = p_a & \text{if } p_b - p_a > \theta(x_b - x_a) \quad \text{(b)} \\ & \text{(Apex captures entire market)} \\ p_a(\frac{1}{2\theta}[(p_b - p_a) + \theta(x_a + x_b)]) & \text{otherwise (the market is divided)} \quad \text{(c)} \end{cases} \tag{22}$$

Brydox has analogous payoffs.

The payoffs result from buyer behavior. A buyer's utility depends on the price he pays and the distance he travels. Price aside, Apex is most attractive of the two sellers to the consumer at $x = 0$ ("consumer 0") and least attractive to the consumer at $x = 1$ ("consumer 1"). Consumer 0 will buy from Apex so long as

$$V - (\theta x_a + p_a) > V - (\theta x_b + p_b), \tag{23}$$

which implies that

$$p_a - p_b < \theta(x_b - x_a), \tag{24}$$

which yields payoff (22a) for Apex. Consumer 1 will buy from Brydox if

$$V - [\theta(1 - x_a) + p_a] < V - [\theta(1 - x_b) + p_b], \tag{25}$$

which implies that

$$p_b - p_a < \theta(x_b - x_a), \tag{26}$$

which yields payoff (22b) for Apex.

Very likely, inequalities (24) and (26) are both satisfied, in which case Consumer 0 goes to Apex and Consumer 1 goes to Brydox. This is the case represented by payoff (22c), and the next task is to find the location of consumer x^*, defined as the consumer who is at the boundary between the two markets, indifferent between Apex and Brydox. First, notice that if Apex attracts Consumer x_b, he also attracts all $x > x_b$, because beyond x_b the consumers' distances from both sellers increase at the same rate. So we know that if there is an indifferent consumer he is between x_a and x_b. Knowing this, the consumer's payoff equation, (21), tells us that

$$V - [\theta(x^* - x_a) + p_a] = V - [\theta(x_b - x^*) + p_b], \tag{27}$$

so that

$$p_b - p_a = \theta(2x^* - x_a - x_b), \tag{28}$$

and

$$x^* = \frac{1}{2\theta}\left[(p_b - p_a) + \theta(x_a + x_b)\right], \tag{29}$$

which generates demand curve (22c)– a differentiated Bertrand demand curve.

Remember, however, that equation (29) is valid only if there really does exist a consumer who is indifferent – if such a consumer does not exist, equation (29) will generates a number for x^*, but that number is meaningless.

Since Apex keeps all the consumers between 0 and x^*, equation (29) is the demand function facing Apex so long as he does not set his price so far above Brydox's that he loses even consumer 0. The demand facing Brydox equals $(1 - x^*)$. Note that if $p_b = p_a$, then from (29), $x^* = \frac{x_a + x_b}{2}$, independent of θ, which is just what we would expect. Demand is linear in the prices of both firms, and looks similar to the demand curves used in Section 3.6 for the Bertrand game with differentiated products.

Now that we have found the demand functions, the Nash equilibrium can be calculated in the same way as in Section 14.2, by setting up the profit functions for each firm, differentiating with respect to the price of each, and solving the two first-order conditions for the two prices. If there exists an equilibrium in which the firms are willing to pick prices to satisfy inequalities (24) and (26), then it is

$$p_a = \frac{(2 + x_a + x_b)\theta}{3}, \quad p_b = \frac{(4 - x_a - x_b)\theta}{3}. \tag{30}$$

From (30) one can see that Apex charges a higher price if a large x_a gives it more safe consumers or a large x_b makes the number of contestable consumers greater. The simplest case is when $x_a = 0$ and $x_b = 1$, when (30) tells us that both firms charge a price equal to θ. Profits are positive and increasing in the transportation cost.

We cannot rest satisfied with the neat equilibrium of equation (30), because the assumption that there exists an equilibrium in which the firms choose prices so as to split the market on each side of some boundary consumer x^* is often violated. Hotelling did not notice this, and fell into a common mathematical trap. Economists are used to models in which the calculus approach gives an answer that is both the local optimum and the global optimum. In games like this one, however, the local optimum is not global, because of the discontinuity in the objective function. Vickrey (1964) and D'Aspremont, Gabszewicz & Thisse (1979) have shown that if x_a and x_b are close together, no pure-strategy equilibrium exists, for reasons similar to why none exists in the Bertrand model with capacity constraints. If both firms charge nonrandom prices, neither would deviate to a slightly different price, but one might deviate to a much lower price that would capture every single consumer. But if both firms charged that low price, each would deviate by raising his price slightly. It turns out that if, for example, Apex and Brydox are located symmetrically around the center of the interval, $x_a \geq 0.25$, and $x_b \leq 0.75$, no pure-strategy equilibrium exists (although a mixed-strategy equilibrium does, as Dasgupta & Maskin [1986b] show).

Hotelling should have done some numerical examples. And he should have thought about the comparative statics carefully. Equation (30) implies that Apex should choose a higher price if both x_a and x_b increase, but it is odd that if the firms are locating closer together, say at 0.90 and 0.91, that Apex should be able to charge a higher price, rather than suffering from more intense competition. This kind of odd result is a typical clue that the result has a logical flaw somewhere. Until the modeller can figure out an intuitive reason for his odd result, he should suspect an error. For practice, let us try a few numerical examples, illustrated in Figure 5.

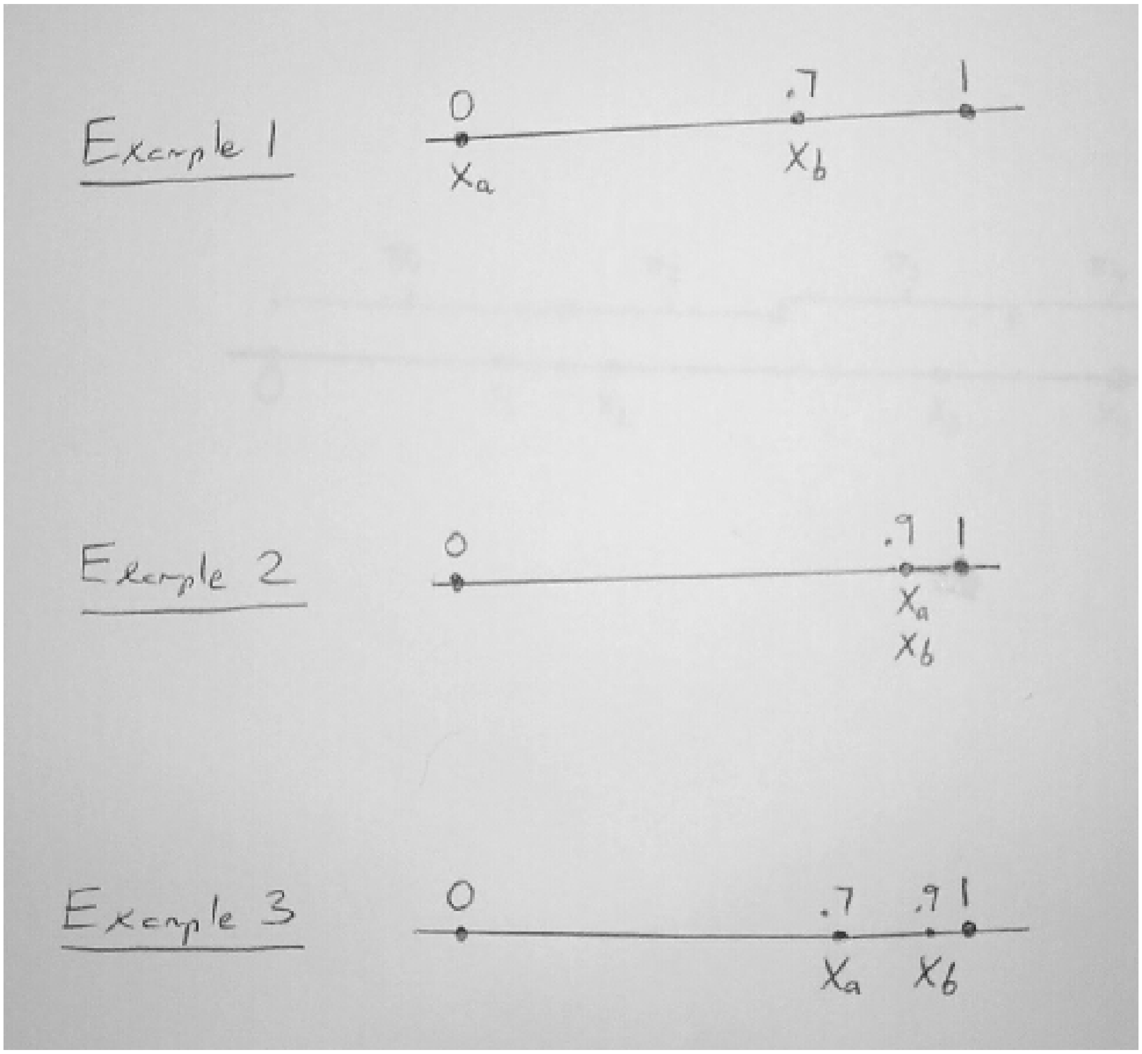

Figure 5: Numerical Examples for Hotelling Pricing

Example 1. Everything works out simply
Try $x_a = 0, x_b = 0.7$ and $\theta = 0.5$. Then equation (30) says $p_a = (2{+}0{+}0.7)0.5/3 = 0.45$ and $p_b = (4{-}0{-}0.7)0.5/3 = 0.55$. Equation (29) says that $x^* = \frac{1}{2*0.5}[(0.55 - 0.45) + 0.5(0.0 + 0.7)] = 0.45$.

In Example 1, there is a pure strategy equilibrium and the equations generated sensible numbers given the parameters we chose. But it is not enough to calculate just one numerical example.

Example 2. Same location – but different prices?
Try $x_a = 0.9, x_b = 0.9$ and $\theta = 0.5$. Then equation (30) says $p_a = (2.0 + 0.9 + 0.9)0.5/3 \approx 0.63$ and $p_b = (4.0 - 0.9 - 0.9)0.5/3 \approx 0.37$.

Example 2 shows something odd happening. The equations generate numbers that seem innocuous until one realizes that if both firms are located at 0.9, but $p_a = 0.63$ and $p_b = 0.37$, then Brydox will capture the entire market! The result is nonsense, because equation (30)'s derivation relied on the assumption that $x_a < x_b$, which is false in this example.

Example 3. Locations too near each other.
$x^* < x_a < x_b$. Try $x_a = 0.7, x_b = 0.9$ and $\theta = 0.5$. Then equation (30) says that $p_a = (2.0 + 0.7 + 0.9)0.5/3 = 0.6$ and $p_b = (4 - 0.7 - 0.9)0.5/3 = 0.4$. As for the split of the market, equation (29) says that $x^* = \frac{1}{2*0.5}[(0.4 - 0.6) + 0.5(0.7 + 0.9)] = 0.6$.

Example 3 shows a serious problem. If the market splits at $x^* = 0.6$ but $x_a = 0.7$ and $x_b = 0.9$, the result violates our implicit assumption that the players split the market. Equation (29) is based on the premise that there does exist some indifferent consumer, and when that is a false premise, as under the parameters of Example 3, equation (29) will still spit out a value of x^*, but the value will not mean anything. In fact the consumer at $x = 0.6$ is not really indifferent between Apex and Brydox. He could buy from Apex at a total cost of $0.6 + 0.1(0.5) = 0.65$ or from Brydox, at a total cost of $0.4 + 0.3\ (0.5) = 0.55$. There exists no consumer who strictly prefers Apex. Even Apex's 'home' consumer at $x = 0.7$ would have a total cost of buying from Brydox of $0.4+0.5(0.9-0.7) = 0.5$ and would prefer Brydox. Similarly, the consumer at $x = 0$ would have a total cost of buying from Brydox of $0.4 + 0.5(0.9 - 0.0) = 0.85$, compared to a cost from Apex of $0.6 + 0.5(0.7 - 0.0) = 0.95$, and he, too, would prefer Brydox.

The problem in Examples 2 and 3 is that the firm with the higher price would do better to deviate with a discontinuous price cut, to just below the other firm's price. Equation (30) was derived by calculus, with the implicit assumption that a local profit maximum was also a global profit maximum, or, put differently, that if no small change could raise a firm's payoff, then it had found the optimal strategy. Sometimes a big change will increase a player's payoff even though a small change would not. Perhaps this is what they mean in business by the importance of "nonlinear thinking" or "thinking out of the envelope." The everyday manager or scientist as described by Schumpeter (1934) and Kuhn (1970) concentrates on analyzing incremental changes and only the entrepreneur or genius breaks through with a discontinuously new idea, the profit source or paradigm shift.

Let us now turn to the choice of location. We will simplify the model by pushing consumers into the background and imposing a single exogenous price on all firms.

The Hotelling Location Game
(Hotelling [1929])

Players
n Sellers.

The Order of Play
The sellers simultaneously choose locations $x_i \in [0, 1]$.

Payoffs
Consumers are distributed along the interval [0,1] with a uniform density equal to one. The

price equals one, and production costs are zero. The sellers are ordered by their location so $x_1 \leq x_2 \leq \ldots \leq x_n$, $x_0 \equiv 0$ and $x_{n+1} \equiv 1$. Seller i attracts half the consumers from the gaps on each side of him, as shown in Figure 14.6, so that his payoff is

$$\pi_1 = x_1 + \frac{x_2 - x_1}{2}, \tag{31}$$

$$\pi_n = \frac{x_n - x_{n-1}}{2} + 1 - x_n, \tag{32}$$

or, for $i = 2, \ldots n-1$,

$$\pi_i = \frac{x_i - x_{i-1}}{2} + \frac{x_{i+1} - x_i}{2}. \tag{33}$$

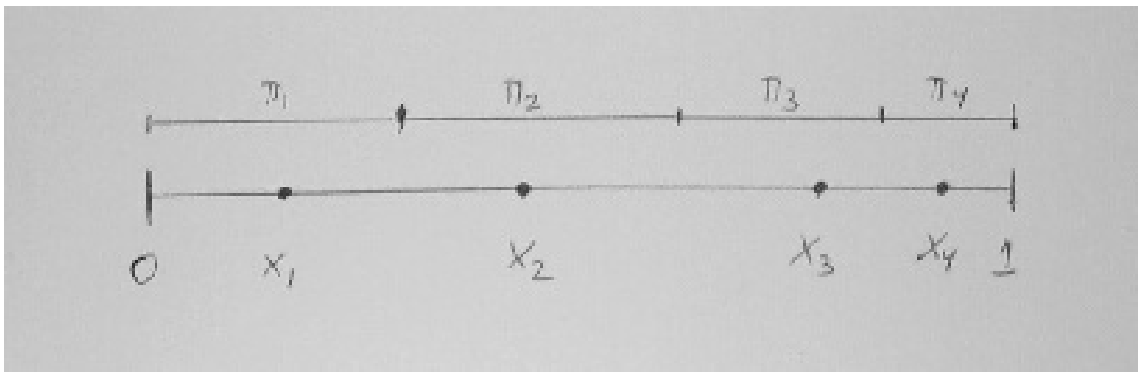

Figure 6: Payoffs in the Hotelling Location Game

With **one seller**, the location does not matter in this model, since the consumers are captive. If price were a choice variable and demand were elastic, we would expect the monopolist to locate at $x = 0.5$.

With **two sellers**, both firms locate at $x = 0.5$, regardless of whether or not demand is elastic. This is a stable Nash equilibrium, as can be seen by inspecting Figure 4 and imagining best responses to each other's location. The best response is always to locate ε closer to the center of the interval than one's rival. When both firms do this, they end up splitting the market since both of them end up exactly at the center.

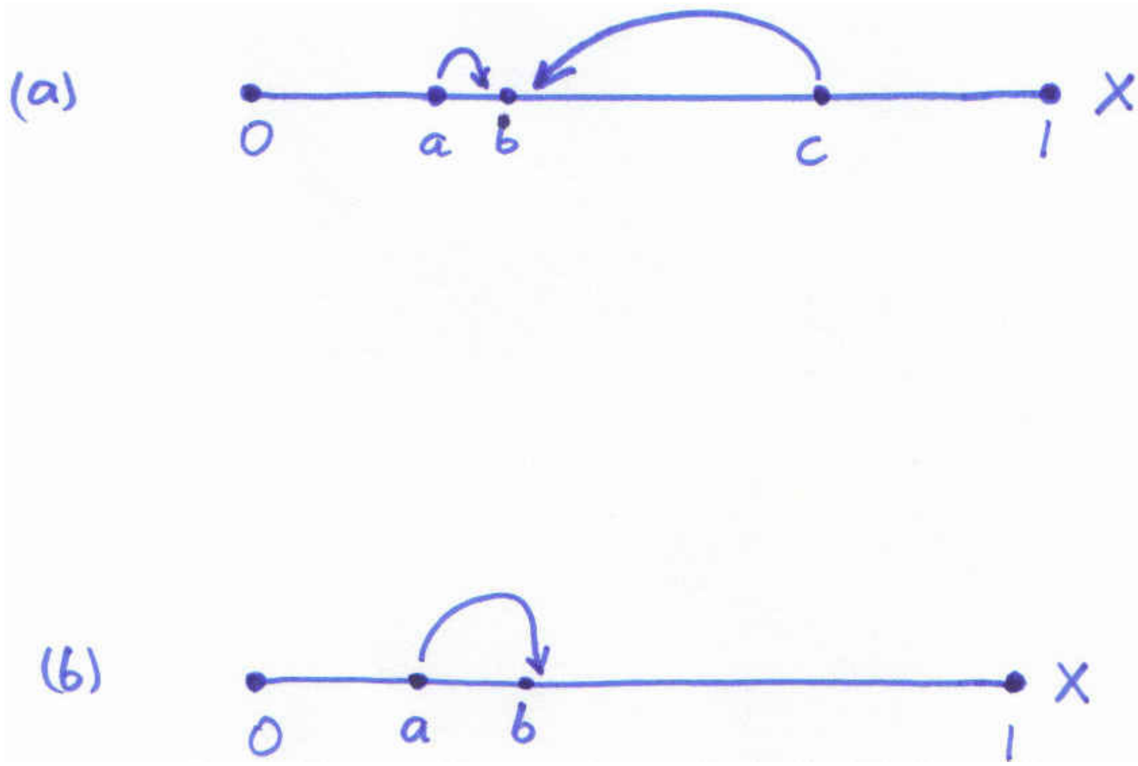

Figure 7: Nonexistence of pure strategies with three players

With **three sellers** the model does not have a Nash equilibrium in pure strategies. Consider any strategy profile in which each player locates at a separate point. Such a strategy profile is not an equilibrium, because the two players nearest the ends would edge in to squeeze the middle player's market share. But if a strategy profile has any two players at the same point a, as in Figure 7, the third player would be able to acquire a share of at least $(0.5 - \epsilon)$ by moving next to them at b; and if the third player's share is that large, one of the doubled-up players would deviate by jumping to his other side and capturing his entire market share. The only equilibrium is in mixed strategies.

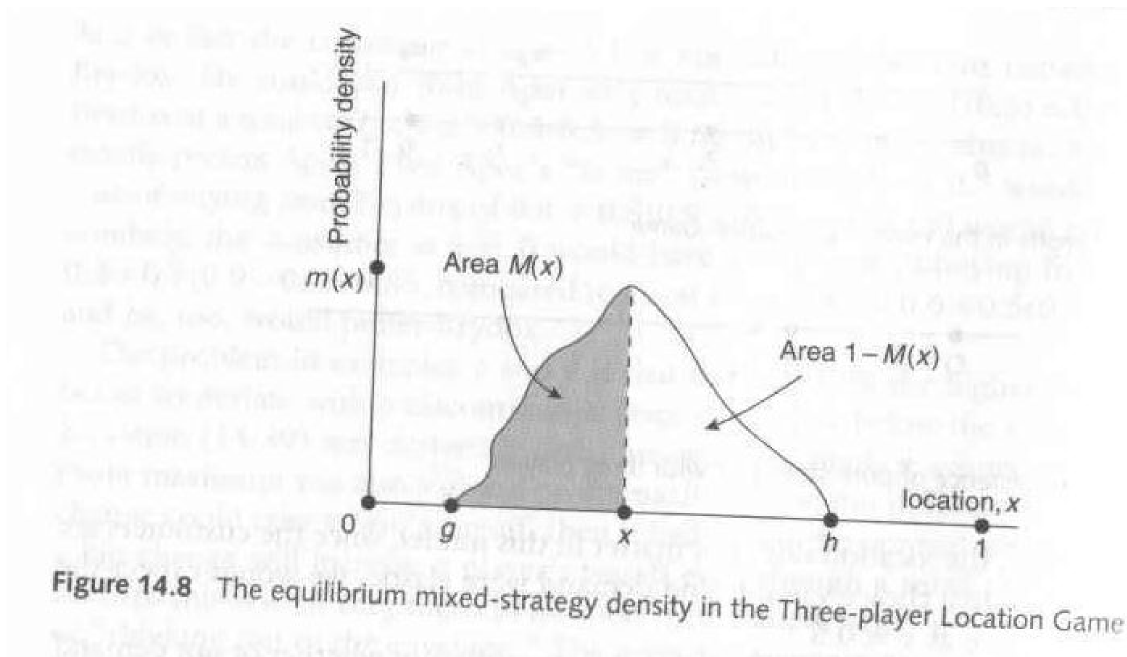

Figure 8: The Equilibrium Mixed-Strategy Density in the Three-Player Location Game

Suppose all three players use the same mixing density, with $m(x)$ the probability density for location x, and positive density on the support $[g, h]$, as depicted in Figure 8. We will need the density for the distribution of the minimum of the locations of Players 2 and 3. Player 2 has location x with density $m(x)$, and Player 3's location is greater than that with probability $1 - M(x)$, letting M denote the cumulative distribution, so the density for Player 2 having location x and it being smaller is $m(x)[1 - M(x)]$. The density for either Player 2 or Player 3 choosing x and it being smaller than the other firm's location is then $2m(x)[1 - M(x)]$.

If Player 1 chooses $x = g$ then his expected payoff is

$$\pi_1(x_1 = g) = g + \int_g^h 2m(x)[1 - M(x)] \left(\frac{x - g}{2}\right) dx, \tag{34}$$

where g is the safe set of consumers to his left, $2m(x)[1 - M(x)]$ is the density for x being the next biggest location of a firm, and $\frac{x-g}{2}$ is Player 1's share of the consumers between his own location of g and the next biggest location.

If Player 1 chooses $x = h$ then his expected payoff is, similarly,

$$\pi_1(x_1 = h) = (1 - h) + \int_g^h 2m(x)M(x) \left(\frac{h - x}{2}\right) dx, \tag{35}$$

where $(1 - h)$ is the set of safe consumers to his right

In a mixed strategy equilibrium, Player 1's payoffs from these two pure strategies must be equal, and they are also equal to his payoff from a location of 0.5, which we can plausibly guess is in the support of his mixing distribution. Going on from this point, the algebra and calculus start to become fierce. Shaked (1982) has computed the symmetric mixing probability density $m(x)$ to be as shown in Figure 9,

$$m(x) = \begin{cases} 2 & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

You can check this equilibrium by seeing that with the mixing density (36) (depicted in Figure 9) the payoffs in equation (34) and (35) do equal each other. This method has only shown what the symmetric equilibrium is like, however; it turns out that asymmetric equilibria also exist (Osborne & Pitchik [1986]).

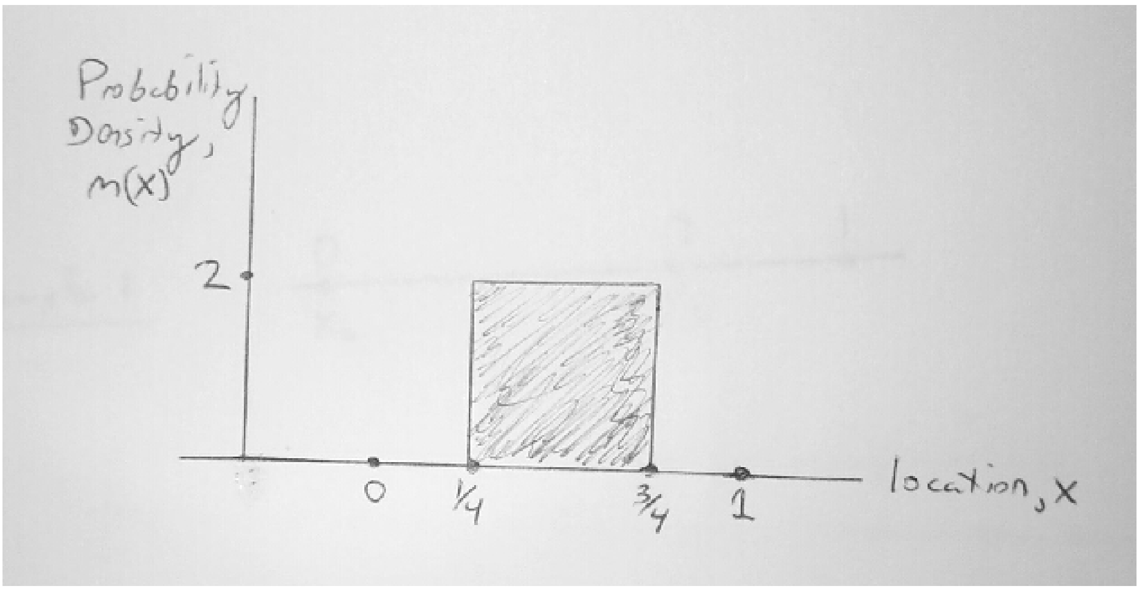

Figure 9: The Equilibrium Mixing Density for Location

Strangely enough, three is a special number. With **more than three sellers**, an equilibrium in pure strategies does exist if the consumers are uniformly distributed, but this is a delicate result (Eaton & Lipsey [1975]). Dasgupta & Maskin (1986b), as amended by Simon (1987), have also shown that an equilibrium, possibly in mixed strategies, exists for any number of players n in a space of any dimension m.

Since prices are inflexible, the competitive market does not achieve efficiency. A benevolent social planner or a monopolist who could charge higher prices if he located his outlets closer to more consumers would choose different locations than competing firms. In particular, when two competing firms both locate in the center of the line, consumers are no better off than if there were just one firm. As shown in Figure 10, the average distance of a consumer from a seller would be minimized by setting $x_1 = 0.25$ and $x_2 = 0.75$, the locations that would be chosen either by the social planner or the monopolist.

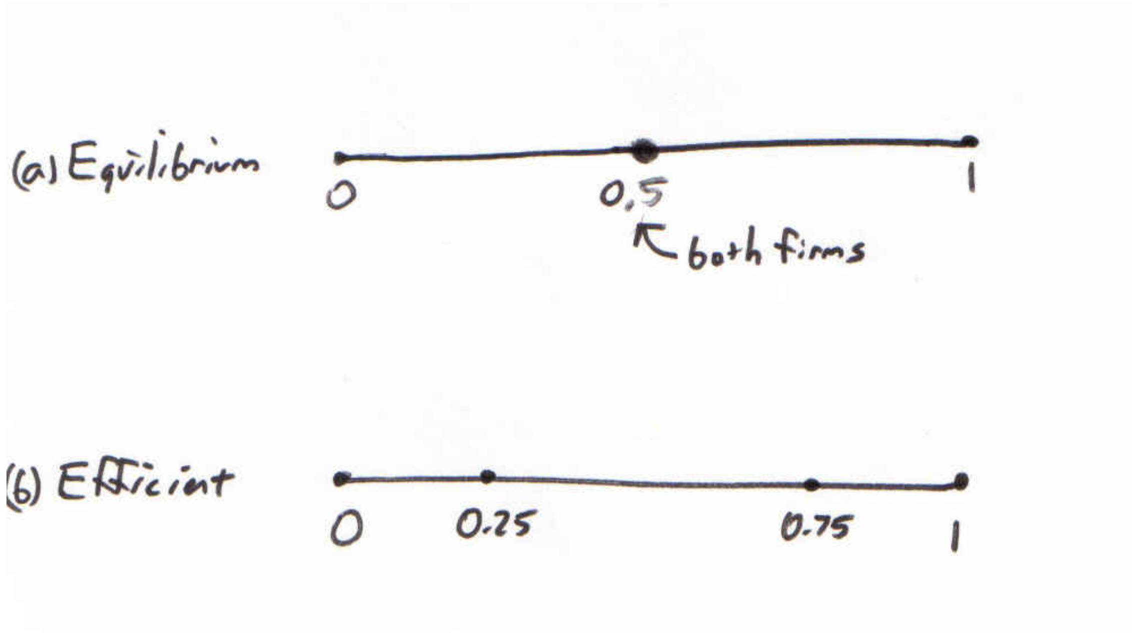

Figure 10: Equilibrium versus Efficiency

The Hotelling Location Model, however, is very well suited to politics. Often there is just one dimension of importance in political races, and voters will vote for the candidate closest to their own position, so there is no analog to price. The Hotelling Location Model predicts that the two candidates will both choose the same position, right on top of the median voter. This seems descriptively realistic; it accords with the common complaint that all politicians are pretty much the same.

14.4 Comparative Statics and Supermodular Games

Comparative statics is the analysis of what happens to endogenous variables in a model when the exogenous variables change. This is a central part of economics. When wages rise, for example, we wish to know how the price of steel will change in response. Game theory presents special problems for comparative statics, because when a parameter changes, not only does Smith's equilibrium strategy change in response, but Jones's strategy changes as a result of Smith's change as well. A small change in the parameter might produce a large change in the equilibrium because of feedback between the different players' strategies.

Let us use a differentiated Bertrand game as an example. Suppose there are N firms, and for firm j the demand curve is

$$Q_j = Max\{\alpha - \beta_j p_j + \sum_{i \neq j} \gamma_i p_i, 0\}, \tag{37}$$

with $\alpha \in (0, \infty)$, $\beta_i \in (0, \infty)$, and $\gamma_i \in (0, \infty)$ for $i = 1, \ldots, N$. Assume that the effect of p_j on firm j's sales is larger than the effect of the other firms' prices, so that

$$\beta_j > \sum_{i \neq j} \gamma_i. \tag{38}$$

Let firm i have constant marginal cost κc_i, where $\kappa \in \{1, 2\}$ and $c_i \in (0, \infty)$, and let us assume that each firm's costs are low enough that it does operate in equilibrium. (The shift variable κ could represent the effect of the political regime on costs.)

The payoff function for firm j is

$$\pi_j = (p_j - \kappa c_j)(\alpha - \beta_j p_j + \sum_{i \neq j} \gamma_i p_i). \tag{39}$$

Firms choose prices simultaneously.

Does this game have an equilibrium? Does it have several equilibria? What happens to the equilibrium price if a parameter such as c_j or κ changes? These are difficult questions because if c_j increases, the immediate effect is to change firm j's price, but the other firms will react to the price change, which in turn will affect j's price. Moreover, this is not a symmetric game – the costs and demand curves differ from firm to firm, which could make algebraic solutions of the Nash equilibrium quite messy. It is not even clear whether the equilibrium is unique.

Two approaches to comparative statics can be used here: the implicit function theorem, and supermodularity. We will look at each in turn.

The Implicit Function Theorem

The implicit-function theorem says that if $f(y, z) = 0$, where y is endogenous and z is exogenous, then

$$\frac{dy}{dz} = -\left(\frac{\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}}\right). \tag{40}$$

It is worth knowing how to derive this. We start with $f(y, z) = 0$, which can be rewritten as $f(y(z), z)) = 0$, since y is endogenous. Using the calculus chain rule,

$$\frac{df}{dz} = \frac{\partial f}{\partial z} + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dz}\right) = 0. \tag{41}$$

where the expression equals zero because after a small change in z, f will still equal zero after y adjusts. Solving for $\frac{dy}{dz}$ yields equation (40).

The implicit function theorem is especially useful if y is a choice variable and z a parameter, because then we can use the first-order condition to set $f(y, z) \equiv \frac{\partial \pi}{\partial y} = 0$ and the second-order condition tells us that $\frac{\partial f}{\partial y} = \frac{\partial^2 \pi}{\partial y^2} \leq 0$. One only has to make certain that the solution is an interior solution, so the first- and second-order conditions are valid, and keep in mind that if the solution is only a local maximum, not a global one, the maximizing choice might "jump" up or down when an exogenous variable changes.

We do have a complication if the model is strategic: there will be more than one endogenous variable, because more than one player is choosing variable values. Suppose that instead of simply $f(y, z) = 0$, our implicit equation has two endogenous and two exogenous variables, so $f(y_1, y_2, z_1, z_2) = 0$. The extra z_2 is no problem; in comparative statics we are holding all but one exogenous variable constant. But the y_2 does add some complexity to the mix. Now, using the calculus chain rule yields not equation (41) but

$$\frac{df}{dz_1} = \frac{\partial f}{\partial z_1} + \left(\frac{\partial f}{\partial y_1}\right)\left(\frac{dy_1}{dz_1}\right) + \left(\frac{\partial f}{\partial y_2}\right)\left(\frac{dy_2}{dz_1}\right) = 0. \tag{42}$$

Solving for $\frac{dy_1}{dz_1}$ yields

$$\frac{dy_1}{dz_1} = -\left(\frac{\frac{\partial f}{\partial z_1} + \left(\frac{\partial f}{\partial y_2}\right)\left(\frac{dy_2}{dz_1}\right)}{\frac{\partial f}{\partial y_1}}\right). \tag{43}$$

It is often unsatisfactory to solve out for $\frac{dy_1}{dz_1}$ as a function of both the exogenous variables z_1 and z_2 and the endogenous variable y_2 (though it is okay if all you want is to discover whether the change is positive or negative), but ordinarily the modeller will also have available an optimality condition for Player 2 also: $g(y_1, y_2, z_1, z_2) = 0$. This second condition yields an equation similar to (43), so that two equations can be solved for the two unknowns.

We can use the differentiated Bertrand game to see how this works out. Equilibrium prices will lie inside the interval $(c_j, \overline{p})$ for some large number $\overline{p}$, because a price of c_j would yield zero profits, rather than the positive profits of a slightly higher price, and $\overline{p}$ can be chosen to yield zero quantity demanded and hence zero profits. The equilibrium or equilibria are, therefore, interior solutions, in which case they satisfy the first-order condition

$$\frac{\partial \pi_j}{\partial p_j} = \alpha - 2\beta_j p_j + \sum_{i \neq j} \gamma_i p_i + \kappa c_j \beta_j = 0, \tag{44}$$

and the second-order condition,

$$\frac{\partial^2 \pi_j}{\partial p_j^2} = -2\beta_j < 0. \tag{45}$$

Next, apply the implicit function theorem by using p_i and c_i, $i = 1, \ldots, N$, instead of y_i and z_i, $i = 1, 2$, and by letting $\frac{\partial \pi_j}{\partial p_j} = 0$ from equation (44) be our $f(y_1, y_2, z_1, z_2) = 0$. The chain rule yields

$$\frac{df}{dc_j} = -2\beta_j \left(\frac{dp_j}{dc_j}\right) + \sum_{i \neq j} \gamma_i \left(\frac{dp_i}{dc_j}\right) + \kappa \beta_j = 0, \tag{46}$$

so

$$\frac{dp_j}{dc_j} = \frac{\sum_{i \neq j} \gamma_i \left(\frac{dp_i}{dc_j}\right) + \kappa \beta_j}{2\beta_j}. \tag{47}$$

Just what is $\frac{dp_i}{dc_j}$? For each i, we need to find the first-order condition for firm i and then use the chain rule again. The first- order condition for Player i is that the derivative of π_i with respect to p_i (*not* p_j) equals zero, so

$$g^i \equiv \frac{\partial \pi_i}{\partial p_i} = \alpha - 2\beta_i p_i + \sum_{k \neq i} \gamma_k p_k + \kappa c_i \beta_i = 0. \tag{48}$$

The chain rule yields (keeping in mind that it is a change in c_j that interests us, *not* a change in c_i),

$$\frac{dg^i}{dc_j} = -2\beta_i \left(\frac{dp_i}{dc_j}\right) + \sum_{k \neq i} \gamma_k \left(\frac{dp_k}{dc_j}\right) = 0. \tag{49}$$

With equation (47), the $(N-1)$ equations (49) give us N equations for the N unknowns $\frac{dp_i}{dc_j}$, $i = 1, \ldots, N$.

It is easier to see what is going on if there are just two firms, j and i. Equations (47) and (49) are then

$$\frac{dp_j}{dc_j} = \frac{\gamma_i \left(\frac{dp_i}{dc_j}\right) + \kappa \beta_j}{2\beta_j}. \tag{50}$$

and

$$-2\beta_i \left(\frac{dp_i}{dc_j}\right) + \gamma_j \left(\frac{dp_j}{dc_j}\right) = 0. \tag{51}$$

Solving these two equations for $\frac{dp_j}{dc_j}$ and $\frac{dp_i}{dc_j}$ yields

$$\frac{dp_j}{dc_j} = \frac{2\beta_i \beta_j \kappa}{4\beta_i \beta_j - \gamma_i \gamma_j} \tag{52}$$

and

$$\frac{dp_i}{dc_j} = \frac{\gamma_j \beta_j \kappa}{4\beta_i \beta_j - \gamma_i \gamma_j}. \tag{53}$$

Keep in mind that the implicit function theorem only tells about infinitesimal changes, not finite changes. If c_n increases enough, then the nature of the equilibrium changes drastically, because firm n goes out of business. Even if c_n increases a finite amount, the implicit function theorem is not applicable, because then the change in p_n will cause changes in the prices of other firms, which will in turn change p_n again.

We cannot go on to discover the effect of changing κ on p_n, because κ is a discrete variable, and the implicit function theorem only applies to continuous variables. The implicit function theorem is none the less very useful when it does apply. This is a simple example, but the approach can be used even when the functions involved are very complicated. In complicated cases, knowing that the second-order condition holds allows the modeller to avoid having to determine the sign of the denominator if all that interests him is the sign of the relationship between the two variables.

Supermodularity

The second approach uses the idea of the supermodular game, an idea related to that of strategic complements (Chapter 3.6). Suppose that there are N players in a game, subscripted by i and j, and that player i has a strategy consisting of $\overline{s}^i$ elements, subscripted by s and t, so his strategy is the vector $y^i = (y^i_1, \ldots, y^i_{\overline{s}^i})$. Let his strategy set be S^i and his payoff function be $\pi^i(y^i, y^{-i}; z)$, where z represents a fixed parameter. We say that the game is a **smooth supermodular game** if the following four conditions are satisfied for every player $i = 1, \ldots N$:

A1′ The strategy set is an interval in $R^{\overline{s}^i}$:

$$S^i = [\underline{y^i}, \overline{y^i}]. \tag{54}$$

A2′ π^i is twice continuously differentiable on S^i.

A3′ (Supermodularity) Increasing one component of player i's strategy does not decrease the net marginal benefit of any other component: for all i, and all s and t such that $1 \leq s < t \leq \overline{s}^i$,

$$\frac{\partial^2 \pi^i}{\partial y^i_s \partial y^i_t} \geq 0. \tag{55}$$

A4′ (Increasing differences in strategies) Increasing one component of j's strategy does not decrease the net marginal benefit of increasing any component of player i's strategy: for all $j \neq i$, and all s and t such that $1 \leq s \leq \overline{s}^i$ and $1 \leq t \leq \overline{s}^j$,

$$\frac{\partial^2 \pi^i}{\partial y^i_s \partial y^j_t} \geq 0. \tag{56}$$

In addition, we will be able to talk about the comparative statics of smooth supermodular games if a fifth condition is satisfied, increasing differences in parameters.

A5′: (Increasing differences in parameters) Increasing parameter z does not decrease the net marginal benefit to player i of any component of his own strategy: for all i, and all s such that $1 \leq s \leq \overline{s}^i$,

$$\frac{\partial^2 \pi^i}{\partial y^i_s \partial z} \geq 0. \tag{57}$$

The heart of supermodularity is in assumptions $A3'$ and $A4'$. Assumption $A3'$ says that the components of player i's strategies are all **complementary inputs**; when one component increases, it is worth increasing the other components too. This means that even if a strategy is a complicated one, one can still arrive at qualitative results about the strategy, because all the components of the optimal strategy will move in the same direction together. Assumption A4′ says that the strategies of players i and j are **strategic complements**; when player i increases a component of his strategy, player j will want to do so also. When the strategies of the players reinforce each other in this way, the feedback between them is less tangled than if they undermined each other.

I have put primes on the assumptions because they are the special cases, for smooth games, of the general definition of supermodular games in the Mathematical Appendix.

Smooth games use differentiable functions, but the supermodularity theorems apply more generally. One condition that is relevant here is condition A5:

A5: (Increasing differences in parameters) π^i has increasing differences in y^i and z for fixed y^{-i}; for all $y^i \geq y^{i\prime}$, the difference $\pi^i(y^i, y^{-i}, z) - \pi^i(y^{i\prime}, y^{-i}, z)$ is nondecreasing with respect to z.

Is the differentiated Bertrand game supermodular? The strategy set can be restricted to $[c_i, \overline{p}]$ for player i, so A1′ is satisfied. π_i is twice continuously differentiable on the interval$[c_i, \overline{p}]$, so A2′ is satisfied. A player's strategy has just one component, p_i, so A3′ is immediately satisfied. The following inequality is true,

$$\frac{\partial^2 \pi_i}{\partial p_i \partial p_j} = \gamma_j > 0, \tag{58}$$

so A4′ is satisfied. And it is also true that

$$\frac{\partial^2 \pi_i}{\partial p_i \partial c_i} = \kappa\beta_i > 0, \tag{59}$$

so A5′ is satisfied for c_i.

From equation (44), $\frac{\partial \pi_i}{\partial p_i}$ is increasing in κ, so $\pi_i(p_i, p_{-i}, \kappa) - \pi_i(p_i', p_{-i}, \kappa)$ is nondecreasing in κ for $p_i > p_i'$, and A5 is satisfied for κ.

Thus, all the assumptions are satisfied. This being the case, a number of theorems can be applied, including the following two.

Theorem 1. *If the game is supermodular, there exists a largest and a smallest Nash equilibrium in pure strategies.*

Theorem 2. *If the game is supermodular and assumption (A5) or (A5′) is satisfied, then the largest and smallest equilibrium are nondecreasing functions of the parameter z.*

Applying Theorems 1 and 2 yields the following results for the differentiated Bertrand game:

1. There exists a largest and a smallest Nash equilibrium in pure strategies (Theorem 1).

2. The largest and smallest equilibrium prices for firm i are nondecreasing functions of the cost parameters c_i and κ (Theorem 2).

Supermodularity, unlike the implicit function theorem, has yielded comparative statics on κ, the discrete exogenous variable. It yields weaker comparative statics on c_i, however, because it just finds the effect of c_i on p_i^* to be nondecreasing, rather than telling us its value or whether it is actually increasing.

For more on supermodularity, see Milgrom & Roberts (1990), Fudenberg & Tirole (1991, pp. 489-497), or Vives's 2005 survey.

*14.5 Vertical Differentiation

In previous sections of this chapter we have been looking at product differentiaton, but differentiation in dimensions that cannot be called "good" versus "bad". Rather, location along a line is a matter of taste and "de gustibus non est disputandum". Another form of product differentiation is from better to worse, as analyzed in Shaked & Sutton (1983) and around pages 150 and 296 of Tirole (1988). Here, we will look at that in a simpler game in which there are just two types of buyers and two levels of quality, but we will compare a monopoly to a duopoly under various circumstances.

Vertical Differentiation I: Monopoly Quality Choice

Players
A seller and a continuum of buyers.

The Order of Play
0 Nature assigns quality values to a continuum of buyers of length 1. Half of them are "weak" buyers ($\theta = 0$) who value high quality at 20 and low quality at 10. Half of them are "strong" buyers ($\theta = 1$) who value high quality at 50 and low quality at 15.
1 The seller picks quality s to be either $s_L = 0$ or $s_H = 1$.
2 The seller picks price p from the interval $[0, \infty)$.
3 Each buyer chooses one unit of a good, or refrains from buying. The seller produces at constant marginal cost $c = 1$, which does not vary with quality.

Payoffs

$$\pi_{seller} = (p-1)q. \tag{60}$$

and

$$\pi_{buyer} = (10 + 5\theta) + (10 + 25\theta)s - p. \tag{61}$$

The seller should clearly set the quality to be high,since then he can charge more to the buyer (though note that this runs contrary to a common misimpression that a monopoly will result in lower quality than a competitive market.) The price should be either 50, which is the most the strong buyers would pay, or 20, the most the weak buyers would pay. Since $\pi(50) = 0.5(50-1) = 24.5$ and $\pi(20) = 0.5(20-1) + 0.5(20-1) = 19$, the seller should choose $p = 50$. Separation (by inducing only the strong buyer to buy) is better for the seller than pooling.

Next we will allow the seller to use two quality levels. A social planner would just use one– the maximal one of $s = s^*$— since it is no cheaper to produce lower quality. The monopoly seller might use two, however, because it helps him to price-discriminate.

Vertical Differentiation II: Crimping the Product

Players
A seller and a continuum of buyers.

The Order of Play
0 Nature assigns quality values to a continuum of buyers of length 1. Half of them are "weak" buyers ($\theta = 0$) who value high quality at 20 and low quality at 10. Half of them are "strong" buyers ($\theta = 1$) who value high quality at 50 and low quality at 15.
1 The seller decides to sell both qualities $s_L = 0$ and $s_H = 1$ or just one of them.
2 The seller picks prices p_L and p_H from the interval $[0, \infty)$.
3 Each buyer chooses one unit of a good, or refrains from buying. The seller produces at constant marginal cost $c = 1$, which does not vary with quality.

Payoffs

$$\pi_{seller} = (p_L - 1)q_L + (p_H - 1)q_H. \tag{62}$$

and

$$\pi_{buyer} = (10 + 5\theta) + (10 + 25\theta)s - p. \tag{63}$$

This is a problem of mechanism design. The seller needs to pick p_1, and p_2 to satisfy incentive compatibility and participation constraints if he wants to offer two qualities with positive sales of both, and he also needs to decide if that is more profitable than offering just one quality.

We already solved the one-quality problem in Vertical Differentiation I, yielding profit of 24.5. The monopolist cannot simply add a second, low- quality, low-price good for the weak buyers, because the strong buyers, who derive zero payoff from the high-quality good, would switch to the low-quality good, which would give them a positive payoff. In equilibrium, the monopolist will have to give the strong buyers a positive payoff. Their participation constraint will be non-binding, as we have found so many times before for the "good" type.

Following the usual pattern, the participation constraint for the weak buyers will be binding, so $p_L = 10$. The self-selection constraint for the strong buyers will also be binding, so

$$\pi_{strong}(L) = 15 - p_L = 50 - p_H. \tag{64}$$

Since $p_L = 10$, this results in $p_H = 45$. The price for high quality must be at least 35 higher than the price for low quality to induce separation of the buyer types.

Profits will now be:

$$\pi_{seller} = (10 - 1)(0.5) + (44 - 1)(0.5) = 26. \tag{65}$$

This exceeds the one-quality profit of 24.5, so it is optimal for the seller to sell two qualities.

This result, is, of course, dependent on the parameters chosen, but it is nonetheless a fascinating special case, and one which is perhaps no more special than the other special case, in which the seller finds that profits are maximized with just one quality. The outcome of allowing price discrimination is a pareto improvement. The seller is better off, because profit has risen from 24.5 to 26. The strong buyers are better off, because the price they pay has fallen from 50 to 45. And the weak buyers are no worse off. In Vertical Differentiation I their payoff was zero because they chose not to buy; in Vertical Differentiation I their payoffs are zero because they buy at a price exactly equal to their value for the good.

Indeed, we can go further. Suppose the cost for the low-quality good was actually *higher* than for the high-quality good, e.g., $p_L = 3$ and $p_H = 1$, because the good is normally produced as high quality and needs to be purposely damaged before it becomes low quality. The price-discrimination profit in (65) would then be $\pi_{seller} = (10-3)(0.5)+(44-1)(0.5) = 25$. Since that is still higher than 24.5, the seller would still price- discriminate. The buyers' payoffs would be unaffected. Thus, allowing the seller to damage some of the good at a cost in real resources of 2 per unit, converting it from high to low quality, can result in a pareto improvement!

This is the point made in Deneckere & McAfee (1996), which illustrates the theory with real-world examples of computer chips and printers purposely damaged to allow price discrimination. See too McAfee (2002, p. 265), which tells us, for example, that Sony made two sizes of minidisc in 2002, a 60-minute and a 74-minute version. Production of both starts with a capacity of 74 minutes, but Sony added code to the 60-minute disc to make 14 minutes of it unusable. That code is an extra fixed cost, but IBM's 1990 Laserprinter E is an example of a damaged product with extra marginal cost. The Laserprinter E was a version of the original Laserprinter that was only half as fast. The reason? IBM added five extra chips to the Laserprinter E to slow it down.

We will analyze one more version of the product differentiation game: with two sellers instead of one. This will show how the product differentiation which increases profits in the way we have seen in the Hotelling games can occur vertically as well as horizontally.

Vertical Differentiation III: Duopoly Quality Choice

Players
Two sellers and a continuum of buyers.

The Order of Play
0 Nature assigns quality values to a continuum of buyers of length 1. Half of them are "weak" buyers ($\theta = 0$) who value high quality at 20 and low quality at 10. Half of them are "strong" buyers ($\theta = 1$) who value high quality at 50 and low quality at 15.
1 Sellers 1 and 2 simultaneously choose values for s_1 and s_2 from the set $\{s_L = 0, s_H = 1\}$. They may both choose the same value.
2 Sellers 1 and 2 simultaneously choose prices p_1 and p_2 from the interval $[0, \infty)$.
3 Each buyer chooses one unit of a good, or refrains from buying. The sellers produce at constant marginal cost $c = 1$, which does not vary with quality.

Payoffs

$$\pi_{seller} = (p - 1)q \tag{66}$$

and

$$\pi_{buyer} = (10 + 5\theta) + (10 + 25\theta)s - p. \tag{67}$$

If both sellers both choose the same quality level, their profits will be zero, but if they choose different quality levels, profits will be positive. Thus, there are three possible equilibria in the quality stage of the game: (Low, High), (High, Low), and a symmetric mixed-strategy equilibrium. Let us consider the pure-strategy equilibria first, and without loss of generality suppose that Seller 1 is the low-quality seller and Seller 2 is the high-quality seller.

(1) The equilibrium prices of Vertical Differentiation II, $(p_L = 10, p_H = 45)$, will no longer be equilibrium prices. The problem is that the low-quality seller would deviate to $p_L = 9$, doubling his sales for a small reduction in price.

(2) Indeed, there is no pure-strategy equilibrium in prices. We have seen that $(p_L = 10, p_H = 45)$ is not an equilibrium, even though $p_H = 45$ is the high- quality seller's best response to $p_L = 10$. $P_L > 10$ will attract no buyers, so that cannot be part of an equilibrium. Suppose $P_L \in (1, 10)$. The response of the high-quality seller will be to set $p_H = p_L + 35$, in which case the low-quality seller can increase his profits by slightly reducing p_L and doubling his sales. The only price left for the low-quality seller that does not generate negative profits is $p_L = 1$, but that yields zero profits, and so is worse than $p_L = 10$. So no choice of p_L is part of a pure-strategy equilibrium.

(3) As always, an equilibrium does exist, so it must be in mixed strategies, as shown below.

The Asymmetric Equilibrium: Pure Strategies for Quality, Mixed for Price

The low-quality seller picks p_L on the support $[5.5, 10]$ using the cumulative distribution

$$F(p_L) = 1 - \left(\frac{39.5}{p_L + 34}\right) \tag{68}$$

with an atom of probability $\frac{39.5}{44}$ at $p_L = 10$.

The high-quality seller picks p_H on the support $[40.5, 45]$ using the cumulative distribution

$$G(p_H) = 2 - \left(\frac{9}{p_H - 36}\right) \tag{69}$$

Weak buyers from the low-quality seller if $10 - p_L \geq 20 - p_H$, which is always true in equilibrium. Strong buyers buy from the low-quality seller if $15 - p_L > 50 - p_H$, which has positive probability, and otherwise from the high-quality seller.

This equilibrium is noteworthy because it includes a probability atom in the mixed-strategy distribution, something not uncommon in pricing games. The low- quality seller usually chooses $p_L = 10$, but with some probability he mixes between 5.5 and 10. The intuition for why this happens is that for the low- quality seller the weak buyers are "safe" customers, for whom the monopoly price is 10, but unless the low-quality seller chooses to shade the price with some probability to try to attract the strong customers, the high-quality seller will maintain such a high price ($p_H = 45$) as to make such shading irresistable.

To start deriving this equilibrium, let us conjecture that the low-quality seller will not include any prices above 10 in his mixing support but will include $p_L = 10$ itself. That is plausible because he would lose all the low-quality buyers at prices above 10, but $p_L = 10$ yields maximal profits whenever p_H is low enough that only weak consumers buy low quality.

The low-quality seller's profit from $p_L = 10$ is $\pi_L(p = 10) = 0.5(10 - 1) = 4.5$. Thus, the lower bound of the support of his mixing distribution (denote it by a_L) must also yield a profit of 4.5. There is no point in charging a price less than the price which would capture even the strong consumers with probability one, in which case

$$\pi_L(a_L) = 0.5(a_L - 1) + 0.5(a_L - 1) = 4.5, \tag{70}$$

and $a_L = 5.5$. Thus, the low-quality seller mixes on $[5.5, 10]$.

On that mixing support, the low-quality seller's profit must equal 4.5 for any price. Thus,

$$\begin{aligned} \pi_L(p_L) = 4.5 \quad &= 0.5(p_L - 1) + 0.5(p_L - 1)Prob(15 - p_L > 50 - p_H) \\ &= 0.5(p_L - 1) + 0.5(p_L - 1)Prob(p_H > 35 + p_L) \\ &= 0.5(p_L - 1) + 0.5(p_L - 1)[1 - G(35 + p_L)] \end{aligned} \tag{71}$$

Thus, the $G(p_H)$ function is such that

$$1 - G(35 + p_L) = \frac{4.5}{0.5(p_L - 1)} - 1 \tag{72}$$

and

$$G(35 + p_L) = 2 - \left(\frac{4.5}{0.5(p_L - 1)}\right). \tag{73}$$

We want a G function with the argument p_H, not $(35 + p_L)$, so let's

$$G(p_H) = 2 - \left(\frac{4.5}{0.5([p_H - 35] - 1)}\right) = 2 - \left(\frac{9}{p_H - 36}\right). \tag{74}$$

As explained in Chapter 3, what we have just done is to find the strategy for the high-quality seller that makes the low-quality seller indifferent among all the values of p_L in his mixing support.

We can find the support of the high-quality seller's mixing distribution by finding values a_H and b_H such that $G(a_H) = 0$ and $G(b_H) = 1$, so

$$G(a_H) = 2 - \left(\frac{9}{a_H - 36}\right) = 0, \tag{75}$$

which yields $a_H = 40.5$, and

$$G(b_H) = 2 - \left(\frac{9}{(0) \cdot b_H - 36}\right) = 1, \tag{76}$$

which yields $b_H = 45$. Thus the support of the high-quality seller's mixing distribution is $[40.5, 45]$.

Now let us find the low-quality seller's mixing distribution, $F(p_L)$. At $p_H = 40.5$, the high-quality seller has zero probability of losing the strong buyers to the low-quality seller, so his profit is $0.5(40.5 - 1) = 19.75$. Now comes the tricky step. At $p_h = 45$, if the high-quality seller had probability one of losing the strong buyers to the low-quality seller, his his profit would be zero, and he would strictly prefer $p_H = 40.5$. Thus, it must be that at $p_h = 45$ there is strictly positive probability that $p_L = 10$— not just a positive density. So let us continue, using our finding that the profit of the high-quality seller must be 19.75 from any price in the mixing support. Then,

$$\begin{aligned} \pi_H(p_H) = 19.75 \quad &= 0.5(p_H - 1)Prob(15 - p_L < 50 - p_H) \\ &= 0.5(p_H - 1)Prob(p_H - 35 < p_L) \\ &= 0.5(p_H - 1)[1 - F(p_H - 35)] \end{aligned} \tag{77}$$

so

$$F(p_H - 35) = 1 - \left(\frac{19.75}{0.5(p_H - 1)}\right). \tag{78}$$

Using the same substitution trick as in equation (74), putting p_L instead of $(p_H - 35)$ as the argument for F, we get

$$F(p_L) \quad = 1 - \left(\tfrac{19.75}{0.5(p_L+35-1)}\right) = 1 - \left(\tfrac{39.5}{p_L+34}\right) \tag{79}$$

In particular, note that

$$F(5.5) = 1 - \left(\frac{39.5}{5.5 + 34}\right) = 0, \tag{80}$$

confirming our earlier finding that the minimum p_L used is 5.5, and

$$F(10) = 1 - \left(\frac{39.5}{10 + 34}\right) = 1 - \frac{39.5}{44} < 1. \tag{81}$$

Equation (81) shows that at the upper bound of the low-quality seller's mixing support the cumulative mixing distribution does not equal 1, an oddity we usually do not see in mixing distributions. What it implies is that there is an atom of probability at $p_L = 10$, soaking up all the remaining probability beyond what equation (81) yields for the prices below 10. The atom must equal $\frac{39.5}{44} \approx 0.9$.

Happily, this solves our paradox of zero high-quality seller profit at $p_H = 45$. If $p_L = 10$ has probability $\frac{39.5}{44}$, the profit from $p_H = 45$ is $0.5(\frac{39.5}{44})(45 - 1) = 19.75$. Thus, the profit from $p_H = 45$ is the same as from $p_H = 40.5$, and the seller is willing to mix between them.

One of the technical lessons of Chapter 3 was that if your attempt to calculate mixing probability results in probabilities of less than zero or more than one, then probably the equilibrium is not in mixed strategies (algebra mistakes being another possibility). The lesson here is that if your attempt to calculate the support of a mixing distribution results in impossible bounds, then you should consider the possibility that the distribution has atoms of probability.

The duopoly sellers' profits are 4.5 (for low-quality) and 19.75 (for high quality) in the asymmetric equilibrium of Vertical Differentiation III, a total of 24.25 for the industry. This is less than either the 24.5 earned by the nondiscriminating monopolist of Vertical Differentiation I or the 26 earned by the discriminating monopolist of Vertical Differentiation II. But what about the mixed-strategy equilibrium for Vertical Differentiation III?

The Symmetric Equilibrium: Mixed Strategies for Both Quality and Price

Each player chooses low quality with probability $\alpha = 4.5/24.25$ and high quality otherwise. If they choose the same quality, they next both choose a price equal to 1, marginal cost. If they choose different qualities, they choose prices according to the mixing distributions in the asymmetric equilibrium.

This equilibrium is easier to explain. Working back from the end, if they choose the same qualities, the two firms are in undifferentiated price competition and will choose prices equal to marginal cost, with payoffs of zero. If they choose different qualities, they are in the same situation as they would be in the asymmetric equilibrium, with expected payoffs of 4.5 for the low-quality firm and 19.75 for the high-quality firm. As for choice of product quality, the expected payoffs from each quality must be equal in equilibrium, so there must be a higher probability of both choosing high-quality:

$$\pi(Low) = \alpha(0) + (1-\alpha)4.5 = \pi(High) = \alpha(19.75) + (1-\alpha)(0). \tag{82}$$

Solving equation (82) yields $\alpha = 4.5/24.25 \approx 0.17$, in which case each player's payoff is about 3.75. Thus, even if a player is stuck in the role of low-quality seller in the pure-strategy equilibrium, with an expected payoff of 4.5, that is better than the expected payoff he would get in the "fairer" symmetric equilibrium.

We can conclude that if the players could somehow arrange what equilibrium would be played out, they would arrange for a pure-strategy equilibrium, perhaps by use of cheap talk and some random focal point variable.

Or, perhaps they could change the rules of the game so that they would choose qualities sequentially. Suppose one seller gets to choose quality first. He would of course choose high quality, for a payoff of 19.75. The second-mover, hwoever, choosing low-quality, would have a payoff of 4.5, better than the expected payoff in the symmetric mixed-strategy equilibrium of the simultaneous quality-choice game. This is the same phenomenon as the pareto superiority of a sequential version of the Battle of the Sexes over the symmetric mixed-strategy equilibrium of the simultaneous-move game.

What if Seller 1 chooses both quality and price first, and Seller 2 responds with quality and price? If Seller 1 chooses low quality, then his optimal price is $p_L = 10$, since the second player will choose high quality and a price low enough to attract the strong buyers— $p_H = 45$, in equilibrium— so Seller 1's payoff would be 0.5(10-1) = 4.5. If Seller 1 chooses high quality, then his optimal price is $p_H = 40.5$, since the second player will choose low quality and would choose a price high enough to lure away the strong buyers if $p_H < 40.5$. If, however, $p_H = 40.5$, Seller 2 would give up on attracting the strong buyers and pick $p_L = 10$. Thus, if Seller 1 chooses both quality and price first, he will choose high quality and $p_H = 40.5$ while Seller 2 will choose low quality and $p_L = 10$, resulting in the same payoffs as in the asymmetric equilibrium of the simultaneous-move game, though no longer in mixed strategies.

What Product Differentiation III shows us is that product differentiation can take place in oligopoly vertically as well as horizontally. Head-to-head competition reduces profits, so firms will try to differentiate in any way that they can. This increases their profits, but it can also benefit consumers— though more obviously in the case of horizontal differentiation than in vertical. Keep in mind, though, that in our games here we have assumed that high quality costs no more than low quality. Usually high quality is more expensive, which means that having more than one quality level can be efficient. Often poor people prefer lower quality, given the cost of higher quality, and even a social planner would provide a variety of quality levels. Here, we see that even when only high quality would be provided in the first-best, it is better that a monopolist provide two qualities than one, and a duopoly is still better for consumers.

*14.6 Durable Monopoly

Introductory economics courses are vague on the issue of the time period over which transactions take place. When a diagram shows the supply and demand for widgets, the x-axis is labelled "widgets," not "widgets per week" or "widgets per year." Also, the diagram splits off one time period from future time periods, using the implicit assumption that supply and demand in one period is unaffected by events of future periods. One problem with this on the demand side is that the purchase of a good which lasts for more than one use is an investment; although the price is paid now, the utility from the good continues into the future. If Smith buys a house, he is buying not just the right to live in the house tomorrow, but the right to live in it for many years to come, or even to live in it for a few years and then sell the remaining years to someone else. The continuing utility he receives from this durable good is called its **service flow**. Even though he may not intend to rent out the house, it is an investment decision for him because it trades off present expenditure for future utility. Since even a shirt produces a service flow over more than an instant of time, the durability of goods presents difficult definitional problems for national income accounts. Houses are counted as part of national investment (and an estimate of their service flow as part of services consumption), automobiles as durable goods consumption, and shirts as nondurable goods consumption, but all are to some extent durable investments.

In microeconomic theory, "durable monopoly" refers not to monopolies that last a long time, but to monopolies that sell durable goods. These present a curious problem. When a monopolist sells something like a refrigerator to a consumer, that consumer drops out of the market until the refrigerator wears out. The demand curve is, therefore, changing over time as a result of the monopolist's choice of price, which means that the modeller should not make his decisions in one period and ignore future periods. Demand is not **time separable**, because a rise in price at time t_1 affects the quantity demanded at time t_2.

The durable monopolist has a special problem because in a sense he does have a competitor – himself in the later periods. If he were to set a high price in the first period, thereby removing high-demand buyers from the market, he would be tempted to set a lower price in the next period to take advantage of the remaining consumers. But if it were expected that he would lower the price, the high-demand buyers would not buy at a high price in the first period. The threat of the future low price forces the monopolist to keep his current price low.

This presents another aspect of product differentiation: the durability of a good. Will a monopolist produce a shoddier, less durable product? Durability is different from the vertical differentiation we have already analyzed because durability has temporal implications. The buyer of a less durable product will return to the market sooner than the buyer of a more durable one, regardless of other aspects of product quality.

To formalize this situation, let the seller have a monopoly on a durable good which lasts two periods. He must set a price for each period, and the buyer must decide what quantity to buy in each period. Because this one buyer is meant to represent the entire market demand, the moves are ordered so that he has no market power, as in the principal-agent models in Chapter 7 and onwards. Alternatively, the buyer can be viewed as representing a continuum of consumers (see Coase [1972] and Bulow [1982]). In this interpretation, instead of "the buyer" buying q_1 in the first period, q_1 of the buyers each buy one unit in the first period.

Durable Monopoly

Players
A buyer and a seller.

The Order of Play
1 The seller picks the first-period price, p_1.
2 The buyer buys quantity q_1 and consumes service flow q_1.
3 The seller picks the second-period price, p_2.
4 The buyer buys additional quantity q_2 and consumes service flow $(q_1 + q_2)$.

Payoffs
Production cost is zero and there is no discounting. The seller's payoff is his revenue, and the buyer's payoff is the sum across periods of his benefits from consumption minus his expenditure. The buyer's benefits arise from his being willing to pay as much as

$$B(q_t) = 60 - \frac{q_t}{2} \tag{83}$$

for the marginal unit service flow consumed in period t, as shown in Figure 10. The payoffs are therefore

$$\pi_{seller} = q_1 p_1 + q_2 p_2 \tag{84}$$

and, since a consumer's total benefit is the sum of a triangle plus a rectangle of benefit, as shown in Figure 10,

$$
\begin{aligned}
\pi_{buyer} &= [consumer\ surplus_1] + [consumer\ surplus_2] \\
&= [total\ benefit_1 - expenditure_1] + [total\ benefit_2 - expenditure_2] \\
&= \left[\left(\frac{(60-B(q_1))q_1}{2} + B(q_1)q_1\right) - p_1q_1\right] \\
&+ \left[\left(\frac{60-B(q_1+q_2)}{2}(q_1+q_2) + B(q_1+q_2)(q_1+q_2)\right) - p_2q_2\right]
\end{aligned}
\tag{85}
$$

Thinking about durable monopoly is hard because we are used to one-period models in which the demand curve, which relates the price to the quantity demanded, is identical to the marginal-benefit curve, which relates the marginal benefit to the quantity consumed. Here, the two curves are different. The marginal benefit curve is the same each period, since it is part of the rules of the game, relating consumption to utility. The demand curve will change over time and depends on the equilibrium strategies, depending as it does on the number of periods left in which to consume the good's services, expected future prices, and the quantity already owned. Marginal benefit is a given for the buyer; quantity demanded is his strategy.

The buyer's total benefit in period 1 is the dollar value of his utility from his purchase of q_1, which equals the amount he would have been willing to pay to rent q_1. This is composed of the two areas shown in Figure 11a, the upper triangle of area $\left(\frac{1}{2}\right)(q_1+q_2)(60-B(q_1+q_2))$ and the lower rectangle of area $(q_1+q_2)B(q_1+q_2)$. From this must be subtracted his expenditure in period 1, p_1q_1, to obtain what we might call his consumer surplus in the first period. Note that p_1q_1 will not be the lower rectangle, unless by some strange accident, and the "consumer surplus" might easily be negative, since the expenditure in period 1 will also yield utility in period 2 because the good is durable.

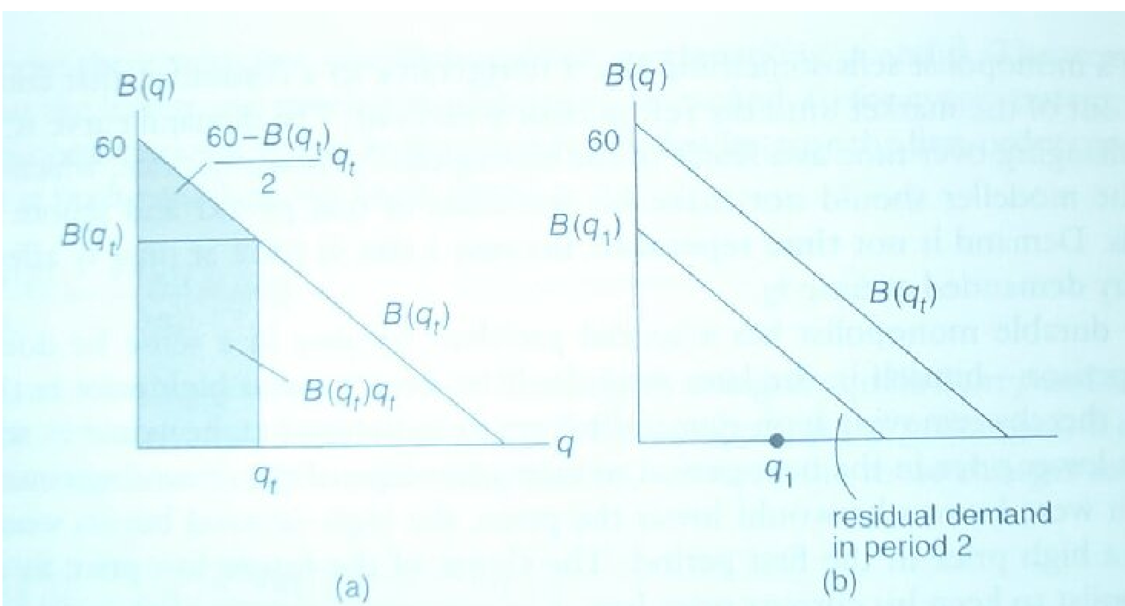

Figure 11: The Buyer's Marginal Benefit per Period in Durable Monopoly

To find the equilibrium price path one cannot simply differentiate the seller's utility with respect to p_1 and p_2, because that would violate the sequential rationality of the seller and the rational response of the buyer. Instead, one must look for a subgame perfect equilibrium, which means starting in the second period and discovering how much the buyer would purchase given his first-period purchase of q_1, and what second-period price the seller would charge given the buyer's second-period demand function.

In the first period, the marginal unit consumed was the $q_1 - th$. In the second period, it will be the $(q_1 + q_2) - th$. The residual demand curve after the first period's purchases is shown in Figure 11b. It is a demand curve very much like the demand curve resulting from intensity rationing in the capacity-constrained Bertrand game of Section 14.2, as shown in Figure 11a. The most intense portion of the buyer's demand, up to q_1 units, has already been satisfied, and what is left begins with a marginal benefit of $B(q_1)$, and falls at the same slope as the original marginal benefit curve. The equation for the residual demand is therefore, using equation (83),

$$p_2 = B(q_1) - \frac{q_2}{2} = 60 - \left(\frac{1}{2}\right) q_1 - \left(\frac{1}{2}\right) q_2. \tag{86}$$

Solving for the monopoly quantity, q_2^*, the seller maximizes $q_2 p_2$, solving the problem

$$\underset{q_2}{Maximize} \; q_2 \left(60 - \left(\frac{1}{2}\right)(q_1 + q_2)\right), \tag{87}$$

which generates the first-order condition

$$60 - q_2 - \left(\frac{1}{2}\right) q_1 = 0, \tag{88}$$

so that

$$q_2^* = 60 - \left(\frac{1}{2}\right) q_1. \tag{89}$$

From equations (86) and (89), it can be seen that $p_2^* = 30 - q_1/4$.

We must now find q_1^*. In period one, the buyer looks ahead to the possibility of buying in period two at a lower price. Buying in the first period has two benefits: consumption of the service flow in the first period and consumption of the service flow in the second period. The price he would pay for a unit in period one cannot exceed the marginal benefit from the first- period service flow in period one plus the foreseen value of p_2, which from (89) is $30 - q_1/4$. If the seller chooses to sell q_1 in the first period, therefore, he can do so at the price

$$\begin{aligned} p_1(q_1) &= B(q_1) + p_2 \\ &= (60 - \left(\tfrac{1}{2}\right) q_1) + (30 - \left(\tfrac{1}{4}\right) q_1), \\ &= 90 - \left(\tfrac{3}{4}\right) q_1. \end{aligned} \tag{90}$$

Knowing that in the second period he will choose q_2 according to (89), the seller combines (89) with (90) to give the maximand in the problem of choosing q_1 to maximize profit over the two periods, which is

$$\begin{aligned}\pi_{seller} = (p_1 q_1 + p_2 q_2) &= \left(90 - \tfrac{3q_1}{4}\right) q_1 + \left(30 - \tfrac{q_1}{4}\right)\left(60 - \tfrac{q_1}{2}\right) \\ &= 1800 + 60q_1 - \tfrac{5q_1^2}{8},\end{aligned} \tag{91}$$

which has the first-order condition

$$60 - \frac{5q_1}{4} = 0, \tag{92}$$

so that

$$q_1^* = 48 \tag{93}$$

and, making use of (90), $p_1^* = 54$.

It follows from (89) that $q_2^* = 36$ and $p_2 = 18$. The seller's profits over the two periods are $\pi_s = 3,240$ $(= 54(48) + 18(36))$.

The purpose of these calculations is to compare the situation with three other market structures: a competitive market, a monopolist who rents instead of selling, and a monopolist who commits to selling only in the first period.

A *competitive market* bids down the price to the marginal cost of zero. Then, $p_1 = 0$ and $q_1 = 120$ from (83) because buyers buy till their marginal benefit is zero, and profits equal zero also.

If the monopolist *rents* instead of selling, then equation (83) is like an ordinary demand equation, because the monopolist is effectively selling the good's services separately each period. He could rent a quantity of 60 each period at a rental fee of 30 and his profits would sum to $\pi_s = 3,600$. That is higher than 3,240, so profits are higher from renting than from selling outright. The problem with selling outright is that the first-period price cannot be very high or the buyer knows that the seller will be tempted to lower the price once the buyer has bought in the first period. Renting avoids this problem.

If the monopolist can *commit to not producing in the second period*, he will do just as well as the monopolist who rents, since he can sell a quantity of 60 at a price of 60, the sum of the rents for the two periods. An example is the artist who breaks the plates for his engravings after a production run of announced size. We must also assume that the artist can convince the market that he has broken the plates. People joke that the best way an artist can increase the value of his work is by dying, and that, too, fits the model.

If the modeller ignored sequential rationality and simply looked for the Nash equilibrium that maximized the payoff of the seller by his choice of p_1 and p_2, he would come to the commitment result. An example of such an equilibrium is ($p_1 = 60$, $p_2 = 200$, *Buyer purchases according to* $q_1 = 120 - p_1$, *and* $q_2 = 0$). This is Nash because neither player has incentive to deviate given the other's strategy, but it fails to be subgame perfect, because the seller should realize that if he deviates and chooses a lower price once the second period is reached, the buyer will respond by deviating from $q_2 = 0$ and will buy more units.

With more than two periods, the difficulties of the durable-goods monopolist become even more striking. In an infinite-period model without discounting, if the marginal cost of production is zero, the equilibrium price for outright sale instead of renting is constant – at zero! Think about this in the context of a model with many buyers. Early consumers foresee that the monopolist has an incentive to cut the price after they buy, in order to sell to the remaining consumers who value the product less. In fact, the monopolist would continue to cut the price and sell more and more units to consumers with weaker and weaker demand until the price fell to marginal cost. Without discounting, even the high-valuation consumers refuse to buy at a high price, because they know they could wait until the price falls to zero. And this is not a trick of infinity: a large number of periods generates a price close to zero.

We can also use the durable monopoly model to think about the durability of the product. If the seller can develop a product so flimsy that it only lasts one period, that is equivalent to renting. A consumer is willing to pay the same price to own a one-hoss shay that he knows will break down in one year as he would pay to rent it for a year. Low durability leads to the same output and profits as renting, which explains why a firm with market power might produce goods that wear out quickly. The explanation is not that the monopolist can use his market power to inflict lower quality on consumers– after all, the price he receives is lower too– but that the lower durability makes it credible to high-valuation buyers that the seller expects their business in the future and will not reduce his price.

With durable-goods monopoly, this book is concluded. Is this book itself a durable good? As I am now writing its fourth edition, I cannot say that it is perfectly durable, because it has improved with each edition, and I can honestly say that a rational consumer who liked the first edition should have bought each successive edition. If you can benefit from this book, your time is valuable enough that you should substitute book reading for solitary thinking even at the expensive prices my publisher and I charge.

Yet although I have added new material, and improved my presentation of the old material, the basic ideas remain the same. The central idea is that in modern economic modelling the modeller starts by thinking about players, actions, information, and payoffs, stripping a situation down to its essentials. Having done that, he sees what payoff-maximizing equilibrium behavior arises from the assumptions. This book teaches a variety of common ways that assumptions link to conclusions just as a book on chess strategy teaches how variety of common configurations of a chessboard lead to winning or losing. Just as with a book on chess, however, the important thing is not just to know common tricks and simplifications, but to be able to recognize the general features of a situation and know what tricks to apply. Chess is just a game, but game theory, I hope, will provide you with tools for improving your life and the policies you recommend to others.

Notes

N14.1 Quantities as Strategies: The Cournot Equilibrium Revisited

- Articles on the existence and uniqueness of a pure-strategy equilibrium in the Cournot model include Roberts & Sonnenschein (1976), Novshek (1985), and Gaudet & Salant (1991).

- **Merger in a Cournot model.** A problem with the Cournot model is that a firm's best policy is often to split up into separate firms. Apex gets half the industry profits in a duopoly game. If Apex split into firms $Apex_1$ and $Apex_2$, it would get two thirds of the profit in the Cournot triopoly game, even though industry profit falls.

  This point was made by Salant, Switzer & Reynolds (1983) and is the subject of problem 14.2. It is interesting that nobody noted this earlier, given the intense interest in Cournot models. The insight comes from approaching the problem from asking whether a player could improve his lot if his strategy space were expanded in reasonable ways.

- An ingenious look at how the number of firms in a market affects the price is Bresnahan & Reiss (1991), which looks empirically at a number of very small markets with one, two, three or more competing firms. They find a big decline in the price from one to two firms, a smaller decline from two to three, and not much change thereafter.

  Exemplifying theory, as discussed in the Introduction to this book, lends itself to explaining particular cases, but it is much less useful for making generalizations across industries. Empirical work associated with exemplifying theory tends to consist of historical anecdote rather than the linear regressions to which economics has become accustomed. Generalization and econometrics are still often useful in industrial organization, however, as Bresnahan & Reiss (1991) shows. The most ambitious attempt to connect general data with the modern theory of industrial organization is Sutton's 1991 book, *Sunk Costs and Market Structure*, which is an extraordinarily well- balanced mix of theory, history, and numerical data.

N14.2 Prices as Strategies: The Bertrand Equilibrium

- As Morrison (1998) points out, Cournot actually does (in Chapter 7) analyze the case of price competition with imperfect substitutes, as well as the quantity competition that bears his name. It is convenient to continue to contrast "Bertrand" and "Cournot" competition, however, though a case can be made for simplifying terminology to "price" and "quantity" competition instead. For the history of how the Bertrand name came to be attached to price competition, see Dimand & Dore (1999).

- Intensity rationing has also been called **efficient rationing**. Sometimes, however, this rationing rule is inefficient. Some low-intensity consumers left facing the high price decide not to buy the product even though their benefit is greater than its marginal cost. The reason intensity rationing has been thought to be efficient is that it is efficient if the rationed-out consumers are unable to buy at any price.

- OPEC has tried both price and quantity controls ("OPEC, Seeking Flexibility, May Choose Not to Set Oil Prices, but to Fix Output," *Wall Street Journal*, October 8, 1987, p. 2; "Saudi King Fahd is Urged by Aides To Link Oil Prices to Spot Markets," *Wall Street Journal*, October 7, 1987, p. 2). Weitzman (1974) is the classic reference on price versus quantity control by regulators, although he does not use the context of oligopoly. The decision rests partly on enforceability, and OPEC has also hired accounting firms to monitor prices ("Dutch Accountants Take On a Formidable Task: Ferreting Out 'Cheaters' in the Ranks of OPEC," *Wall Street Journal*, February 26, 1985, p. 39).

- Kreps & Scheinkman (1983) show how capacity choice and Bertrand pricing can lead to a Cournot outcome. Two firms face downward-sloping market demand. In the first stage of the game, they simultaneously choose capacities, and in the second stage they simultaneously choose prices (possibly by mixed strategies). If a firm cannot satisfy the demand facing it in the second stage (because of the capacity limit), it uses intensity rationing (the results depend on this). The unique subgame perfect equilibrium is for each firm to choose the Cournot capacity and price.

- Haltiwanger & Waldman (1991) have suggested a dichotomy applicable to many different games between players who are **responders**, choosing their actions flexibly, and those who are **nonresponders**, who are inflexible. A player might be a nonresponder because he is irrational, because he moves first, or simply because his strategy set is small. The categories are used in a second dichotomy, between games exhibiting **synergism**, in which responders choose to do whatever the majority do (upward sloping reaction curves), and games exhibiting **congestion**, in which responders want to join the minority (downward sloping reaction curves). Under synergism, the equilibrium is more like what it would be if all the players were nonresponders; under congestion, the responders have more influence. Haltiwanger and Waldman apply the dichotomies to network externalities, efficiency wages, and reputation.

- There are many ways to specify product differentation. This chapter looks at horizontal differentiation where all consumers agree that products A and B are more alike than A and C, but they disagree as to which is best. Another way horizontal differentiation might work is for each consumer to like a particular product best, but to consider all others as equivalent. See Dixit & Stiglitz (1977) for a model along those lines. Or, differentiation might be vertical: all consumers agree that A is better than B and B is better than C but they disagree as to how *much* better A is than B. Firms therefore offer different qualities at different prices. Shaked & Sutton (1983) have explored this kind of vertical differentation.

N14.3 Location models

- For a booklength treatment of location models, see Greenhut & Ohta (1975).

- Vickrey notes the possible absence of a pure-strategy equilibrium in Hotelling's model in pp.323-324 of his 1964 book *Microstatics*. D'Aspremont, Gabszewicz & Thirse (1979) work out the mixed-strategy equilibrium for the case of quadratic transportation costs, and Osborne & Pitchik (1987) do the same for Hotelling's original model.

- Location models and switching cost models are attempts to go beyond the notion of a market price. Antitrust cases are good sources for descriptions of the complexities of pricing in particular markets. See, for example, Sultan's 1974 book on electrical equipment in the 1950s, or antitrust opinions such as *US v. Addyston Pipe & Steel Co.*, 85 F. 271 (1898).

- It is important in location models whether the positions of the players on the line are moveable. See, for example, Lane (1980).

- The location games in this chapter model use a one-dimensional space with end points, i.e., a line segment. Another kind of one-dimensional space is a circle (not to be confused with a disk). The difference is that no point on a circle is distinctive, so no consumer preference can be called extreme. It is, if you like, Peoria versus Berkeley. The circle might be used for modelling convenience or because it fits a situation: e.g., airline flights spread over the 24 hours of the day. With two players, the Hotelling location game on a circle has a continuum of pure-strategy equilibria that are one of two types: both players locating at the same spot, versus players separated from each other by 180°. The three-player model also has a continuum of pure-strategy equilibria, each player separated from another by 120°, in contrast to the nonexistence of a pure-strategy equilibrium when the game is played on a line segment.

- Characteristics such as the color of cars could be modelled as location, but only on a player-by-player basis, because they have no natural ordering. While Smith's ranking of (red=1, yellow=2, blue=10) could be depicted on a line, if Brown's ranking is (red=1, blue=5, yellow=6) we cannot use the same line for him. In the text, the characteristic was something like physical location, about which people may have different preferences but agree on what positions are close to what other positions.

N14.6 Durable Monopoly

- The proposition that price falls to marginal cost in a durable monopoly with no discounting and infinite time is called the "Coase Conjecture," after Coase (1972). It is really a proposition and not a conjecture, but alliteration was too strong to resist.

- Gaskins (1974) has written a well-known article on the problem of the durable monopolist who foresees that he will be creating his own future competition in the future because his product can be recycled, using the context of the aluminum market.

- Leasing by a durable monopoly was the main issue in the antitrust case *US v. United Shoe Machinery Corporation*, 110 F. Supp. 295 (1953), but not because it increased monopoly profits. The complaint was rather that long-term leasing impeded entry by new sellers of shoe machinery, a curious idea when the proposed alternative was outright sale. More likely, leasing was used as a form of financing for the machinery consumers; by leasing, they did not need to borrow as they would have to do if it was a matter of financing a purchase. See Wiley, Ramseyer, and Rasmusen (1990).

- Another way out of the durable monopolist's problem is to give best-price guarantees to consumers, promising to refund part of the purchase price if any future consumer gets a lower price. Perversely, this hurts consumers, because it stops the seller from being tempted to lower his price. The "most-favored- consumer" contract, which is the analogous contract in markets with several sellers, is analyzed by Holt & Scheffman (1987), for example, who demonstrate how it can maintain high prices, and Png & Hirshleifer (1987), who show how it can be used to price discriminate between different types of buyers.

- The durable monopoly model should remind you of bargaining under incomplete information. Both situations can be modelled using two periods, and in both situations the problem for the seller is that he is tempted to offer a low price in the second period after having offered a high price in the first period. In the durable monopoly model this would happen if the high-valuation buyers bought in the first period and thus were absent from consideration by the second period. In the bargaining model this would happen if the buyer rejected the first-period offer and the seller could conclude that he must have a low valuation and act accordingly in the second period. With a rational buyer, neither of these things can happen, and the models' complications arise from the attempt of the seller to get around the problem.

In the durable-monopoly model this would happen if the high-valuation buyers bought in the first period and thus were absent from consideration by the second period. In the bargaining model this would happen if the buyer rejected the first-period offer and the seller could conclude that he must have a low valuation and act accordingly in the second period. For further discussion, see the survey by Kennan & Wilson (1993).

Problems

14.1. Differentiated Bertrand with Advertising (medium)
Two firms that produce substitutes are competing with demand curves

$$q_1 = 10 - \alpha p_1 + \beta p_2 \tag{94}$$

and

$$q_2 = 10 - \alpha p_2 + \beta p_1. \tag{95}$$

Marginal cost is constant at $c = 3$. A player's strategy is his price. Assume that $\alpha > \beta/2$.

(a) What is the reaction function for firm 1? Draw the reaction curves for both firms.

(b) What is the equilibrium? What is the equilibrium quantity for firm 1?

(c) Show how firm 2's reaction function changes when β increases. What happens to the reaction curves in the diagram?

(d) Suppose that an advertising campaign could increase the value of β by one, and that this would increase the profits of each firm by more than the cost of the campaign. What does this mean? If either firm could pay for this campaign, what game would result between them?

14.2. Cournot Mergers (easy) (See Salant, Switzer, & Reynolds [1983])
There are three identical firms in an industry with demand given by $P = 1 - Q$, where $Q = q_1 + q_2 + q_3$. The marginal cost is zero.

(a) Compute the Cournot equilibrium price and quantities.

(b) How do you know that there are no asymmetric Cournot equilibria, in which one firm produces a different amount than the others?

(c) Show that if two of the firms merge, their shareholders are worse off.

14.3. Differentiated Bertrand (medium)
Two firms that produce substitutes have the demand curves

$$q_1 = 1 - \alpha p_1 + \beta(p_2 - p_1) \tag{96}$$

and

$$q_2 = 1 - \alpha p_2 + \beta(p_1 - p_2), \tag{97}$$

where $\alpha > \beta$. Marginal cost is constant at c, where $c < 1/\alpha$. A player's strategy is his price.

(a) What are the equations for the reaction curves $p_1(p_2)$ and $p_2(p_1)$? Draw them.

(b) What is the pure-strategy equilibrium for this game?

(c) What happens to prices if α, β, or c increase?

(d) What happens to each firm's price if α increases, but only firm 2 realizes it (and firm 2 knows that firm 1 is uninformed)? Would firm 2 reveal the change to firm 1?

Problem 14.4. Asymmetric Cournot Duopoly (easy)
Apex has variable costs of q_a^2 and a fixed cost of 1000, while Brydox has variables costs of $2q_b^2$ and no fixed cost. Demand is $p = 115 - q_a - q_b$.

(a) What is the equation for Apex's Cournot reaction function?

(b) What is the equation for Brydox' Cournot reaction function?

(c) What are the outputs and profits in the Cournot equilibrium?

Problem 14.5. Price Discrimination (medium)
A seller faces a large number of buyers whose market demand is given by $P = \alpha - \beta Q$. Production marginal cost is constant at c.

(a) What is the monopoly price and profit?

(b) What are the prices under perfect price discrimination if the seller can make take-it-or-leave-it offers? What is the profit?

(c) What are the prices under perfect price discrimination if the buyer and sellers bargain over the price and split the surplus evenly? What is the profit?

The Kleit Oligopoly Game: A Classroom Game for Chapter 14

The widget industry in Smallsville has N firms. Each firm produces 150 widgets per month. All costs are fixed, because labor is contracted for on a yearly basis, so we can ignore production cost for the purposes of this case. Widgets are perishable; if they are not sold within the month, they explode in flames.

There are two markets for widgets, the national market, and the local market. The price in the national market is $20 per widget, with the customers paying for delivery, but the price in the local market depends on how many are for sale there in a given month. The price is given by the following market demand curve:

$$P = 100 - \frac{Q}{N},$$

where Q is the total output of widgets sold in the local market. If, however, this equation would yield a negative price, the price is just zero, since the excess widgets can be easily destroyed.

$20 is the opportunity cost of selling a widget locally– it is what the firm loses by making that decision. The benefit from the decision depends on what other firms do. All firms make their decisions at the same time on whether to ship widgets out of town to the national market. The train only comes to Smallsville once a month, so firms cannot retract their decisions. If a firm delays making its decision till too late, then it misses the train, and all its output will have to be sold in Smallsville.

General Procedures

For the first seven months, each of you will be a separate firm. You will write down two things on an index card: (1) the number of the month, and (2) your LOCAL-market sales for that month. Also record your local and national market sales on your Scoresheet. The instructor will collect the index cards and then announce the price for that month. You should then calculate your profit for the month and add it to your cumulative total, recording both numbers on your Scoresheet.

For the last five months, you will be organized into five different firms. Each firm has a capacity of 150, and submits a single index card. The card should have the number of the firm on it, as well as the month and the local output. The instructor will then calculate the market price, rounding it to the nearest dollar to make computations easier. Your own computations will be easier if you pick round numbers for your output.

If you do not turn in an index card by the deadline, you have missed the train and all 150 of your units must be sold locally. You can change your decision up until the deadline by handing in a new card noting both your old and your new output, e.g., "I want to change from 40 to 90."

Procedures Each Month

1. Each student is one firm. No talking.
2. Each student is one firm. No talking.
3. Each student is one firm. No talking.
4. Each student is one firm. No talking.

5. Each student is one firm. No talking.

6. Each student is one firm. You can talk with each other, but then you write down your own output and hand all outputs in separately.

7. Each student is one firm. You can talk with each other, but then you write down your own output and hand all outputs in separately.

8. You are organized into Firms 1 through 5, so N=5. People can talk within the firms, but firms cannot talk to each other. The outputs of the firms are secret.

9. You are organized into Firms 1 through 5, so N=5. People can talk within the firms, but firms cannot talk to each other. The outputs of the firms are secret.

10. You are organized into Firms 1 through 5, so N=5. You can talk to anyone you like, but when the talking is done, each firm writes down its output secretly and hands it in.

11. You are organized into Firms 1 through 5, so N=5. You can talk to anyone you like, but when the talking is done, each firm writes down its output secretly and hands it in. Write the number of your firm with your output. This number will be made public once all the outputs have been received.

You may be wondering about the "Kleit". Andrew Kleit is an economics professor Pennsylvania State University who originated the ancestor of this oligopoly game game for classroom use.