PROBLEM SET 1 ANSWERS

12 September 2006.

1.1. Nash and Iterated Dominance (medium)

(a) Show that every iterated dominance equilibrium s^* is Nash.

<u>*Answer.*</u> Suppose that s^* is not Nash. This means that there exists some i and s_i' such that i could profitably deviate, i.e., $\pi_i(s^*) < \pi_i(s_i', s_{-i}^*)$. But that means that there is no point during the iterated deletion at which player i could have eliminated strategy s_i' as being even weakly dominated for him by s_i^*. Hence, iterated deletion could not possibly reach s^* and we have a contradiction; it must be that every iterated dominance equilibrium is Nash.

(b) Show by counterexample that not every Nash equilibrium can be generated by iterated dominance.

<u>*Answer.*</u> In Ranked Coordination (Table 7 of Chapter 1) no strategy can be eliminated by dominance, and the boldfaced strategies are Nash.

(c) Is every iterated dominance equilibrium made up of strategies that are not weakly dominated?

<u>*Answer.*</u> Yes. As defined in Chapter 1, strategy x is weakly dominated by strategy y only if y has a strictly higher payoff in some strategy profile and has a strictly lower payoff in no strategy profile. An iterated dominance equilibrium only exists if the iterative process results in a single strategy profile at the end.

Denote the strategies for player i that are still surviving when there are n stages of iterated deletion still to go before we reach a particular iterated dominance equilibrium by $S_i(n)$.

In the equilibrium, denote player i's strategy by z, so $S_i(0) = \{z\}$. In order for x to be in the final surviving profile, it would have to weakly dominate the next-to-last surviving strategy for that player, w (where $S_i(1) = \{z, w\}$ in our notation). Thus, even in the original game it must be that w does not weakly dominate z.

More generally, strategy v must be weakly dominated by some strategy in $S_i(n)$ to be deleted at stage $(n-1)$. If it is, though, then it cannot dominate that strategy, which is itself weakly dominated by some other strategy in $S_i(n)$ except if $n = 0$. This means in turn that it cannot dominate strategy z, the strategy in the iterated-dominance equilibrium. Q.E.D.

If we define things a bit differently, we would get different answers to this question. Suppose we define "quasi-weakly dominating strategy" as a strategy that is no worse than any other strategy.

A strategy that is in the equilibrium strategy profile might be a bad reply to some strategies that iterated deletion of quasi-dominated strategies removed from the original game. Consider the Iteration Path Game in Table A1.1. The strategy profile (r_1, c_1) is a iterated quasi-dominance equilibrium. Delete r_2, which is weakly dominated by r_1 (0,0,0 is beaten by 2,1,1). Then delete c_3, which is now quasi-weakly dominated by c_1 (12,11 is equal to 12,11). Then delete r_3, which is weakly dominated by r_1 (0,1 is beaten by 2,1). Then delete c_2, which is strongly dominated by c_1 (10 is beaten by 12).

		Column		
		c_1	c_2	c_3
	r_1	**2,12**	1,10	1,12
Row:	r_2	0,7	0,10	0,12
	r_3	0,11	1,10	0,11

Payoffs to: (Row, Column)

Table A1.1: The Iteration Path Game

1.11. A Sequential Prisoner's Dilemma (hard)

Suppose Row moves first, then Column, in the Prisoner's Dilemma. What are the possible actions? What are the possible strategies? Construct a normal form, showing the relationship between strategy profiles and payoffs.

Hint: The normal form is *not* a two-by-two matrix here.

<u>*Answer.*</u> The possible actions are $Confess$ and $Deny$ for each player.

For Column, the strategy set is:

$$\left\{ \begin{array}{c} (C|C,\ C|D), \\ (C|C,\ D|D), \\ (D|C,\ D|D), \\ (D|C,\ C|D) \end{array} \right\}$$

For Row, the strategy set is simply $\{C, D\}$.

The normal form is:

		Column			
		$(C\|C,\ C\|D)$	$(C\|C,\ D\|D)$	$(D\|C,\ D\|D)$	$(D\|C,\ C\|D)$
Row	*Deny*	-10,0	-1, -1	-1, -1	-10,0
	Confess	-8,-8	-8, -8	0, -10	0, -10

Payoffs to: (Row, Column)

The question did not ask for the Nash equilibrium, but it is disappointing not to know it after all that work, so here it is:

Equilibrium	**Strategies**	**Outcome**
E_1	{*Confess*, ($C\|C$) ($C\|D$)}	Both pick $Confess$.

2.3. Cancer Tests (easy) (adapted from McMillan [1992, p. 211])

Imagine that you are being tested for cancer, using a test that is 98 percent accurate. If you indeed have cancer, the test shows positive (indicating cancer) 98 percent of the time. If you do not have cancer, it shows negative 98 percent of the time. You have heard that 1 in 20 people in the population actually have cancer. Now your doctor tells you that you tested positive, but you shouldn't worry because his last 19 patients all died. How worried should you be? What is the probability you have cancer?

<u>*Answer.*</u> Doctors, of course, are not mathematicians. Using Bayes' Rule:

$$\begin{aligned} Prob(Cancer|Positive) &= \frac{Prob(Positive|Cancer)Prob(Cancer)}{Prob(Positive)} \\ &= \frac{0.98(0.05)}{0.98(0.05)+0.02(0.95)} \\ &\approx 0.72. \end{aligned} \tag{1}$$

With a 72 percent chance of cancer, you should be very worried. But at least it is not 98 percent.

Here is another way to see the answer. Suppose 10,000 tests are done. Of these, an average of 500 people have cancer. Of these, 98% test positive on average— 490 people. Of the 9,500 cancer-free people, 2% test positive on average—190 people. Thus there are 680 positive tests, of which 490 are true positives. The probability of having cancer if you test positive is 490/680, about 72% .

This sort of analysis is one reason why HIV testing for the entire population, instead of for high-risk subpopulations, would not be very informative— there would be more false positives than true positives.

It is interesting to see what happens when the numbers change. Suppose the test is only 60% accurate for positives and negatives. Using Bayes' Rule:

$$
\begin{aligned}
Prob(Cancer|Positive) &= \frac{Prob(Positive|Cancer)Prob(Cancer)}{Prob(Positive)} \\
&= \frac{0.60(0.05)}{0.60(0.05)+0.40(0.95)} \qquad (2) \\
&\approx 0.07.
\end{aligned}
$$

Thus, if you test positive, your chances of having cancer go up by about 50% compared to the prior of .05, but you still probably don't have cancer.

2.5. Joint Ventures (medium)

Software Inc. and Hardware Inc. have formed a joint venture. Each can exert either high or low effort, which is equivalent to costs of 20 and 0. Hardware moves first, but Software cannot observe his effort. Revenues are split equally at the end, and the two firms are risk neutral. If both firms exert low effort, total revenues are 100. If the parts are defective, the total revenue is 100; otherwise, if both exert high effort, revenue is 200, but if only one player does, revenue is 100 with probability 0.9 and 200 with probability 0.1. Before they start, both players believe that the probability of defective parts is 0.7. Hardware discovers the truth about the parts by observation before he chooses effort, but Software does not.

(a) Draw the extensive form and put dotted lines around the information sets of Software at any nodes at which he moves.

<u>*Answer.*</u> See Figure A2.1. To understand where the payoff numbers come from, see the answer to part (b).

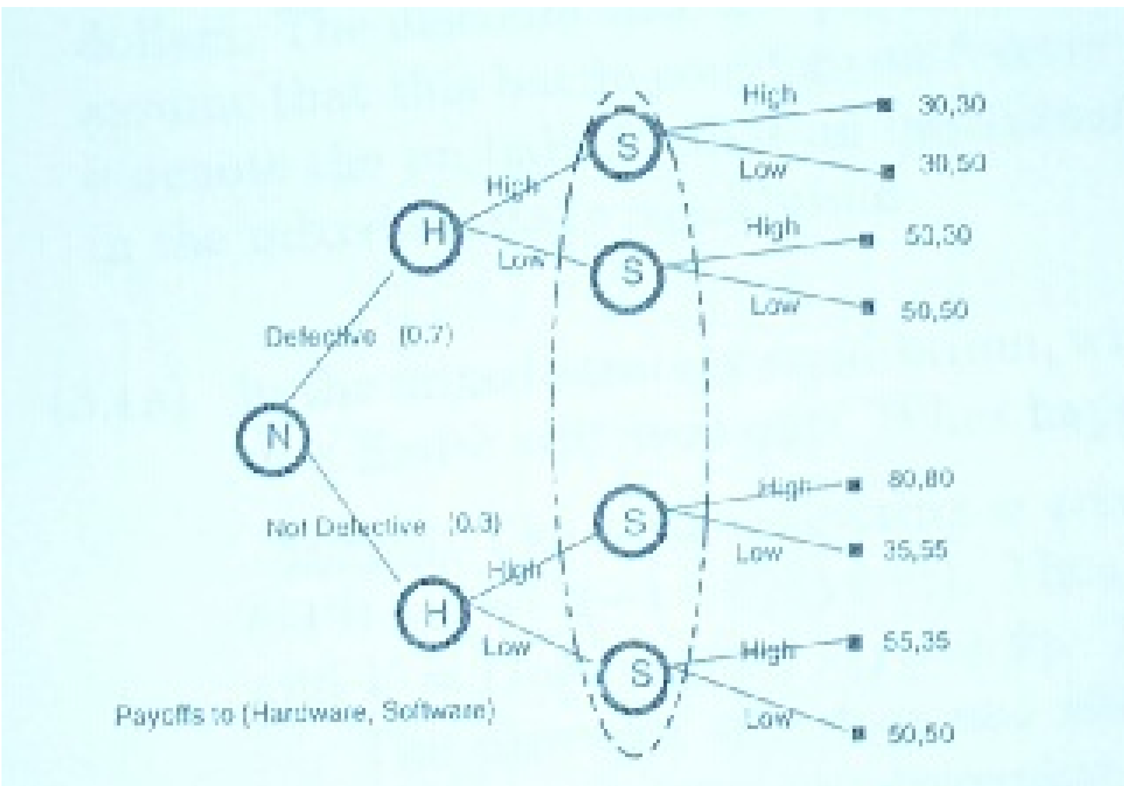

Figure A2.1: The Extensive Form for the Joint Ventures Game

(b) What is the Nash equilibrium?

<u>*Answer.*</u> (Hardware: *Low* if defective parts, *Low* if not defective parts; Software: *Low*).

$$\pi_{Hardware}(Low|Defective) = \frac{100}{2} = 50.$$

Deviating would yield Hardware a lower payoff:

$$\pi_{Hardware}(High|Defective) = \frac{100}{2} - 20 = 30.$$

$$\pi_{Hardware}(Low|Not\ Defective) = \frac{100}{2} = 50.$$

Deviating would yield Hardware a lower payoff:

$$\pi_{Hardware}(High|Not\ Defective) = 0.9\left(\frac{100}{2}\right) + 0.1\left(\frac{200}{2}\right) - 20 = 45 + 10 - 20 = 35.$$

$$\pi_{Software}(Low) = \frac{100}{2} = 50.$$

Deviating would yield Software a lower payoff:

$$\pi_{Software}(High) = 0.7\left(\frac{100}{2}\right) + .3\left[0.9\left(\frac{100}{2}\right) + 0.1\left(\frac{200}{2}\right)\right] - 20 = 35 + 0.3(45+10) - 20.$$

This equals $15 + 0.3(35) = 31.5$, less than the equilibrium payoff of 50.

Elaboration. A strategy combination that is *not* an equilibrium (because Software would deviate) is: (Hardware: *Low* if defective parts, *High* if not defective parts; Software: *High*).

$$\pi_{Hardware}(Low|Defective) = \frac{100}{2} = 50.$$

Deviating would indeed yield Hardware a lower payoff:

$$\pi_{Hardware}(High|Defective) = \frac{100}{2} - 20 = 30.$$

$$\pi_{Hardware}(High|Not\ Defective) = \frac{200}{2} - 20 = 100 - 20 = 80.$$

Deviating would indeed yield Hardware a lower payoff:

$$\pi_{Hardware}(Low|Not\ Defective) = 0.9\left(\frac{100}{2}\right) + 0.1\left(\frac{200}{2}\right) = 55.$$

$$\pi_{Software}(High) = 0.7\left(\frac{100}{2}\right) + 0.3\left(\frac{200}{2}\right) - 20 = 35 + 30 - 20 = 45.$$

Deviating would yield Software a higher payoff, so the strategy combination we are testing is not a Nash equilibrium:

$$\pi_{Software}(Low) = 0.7\left(\frac{100}{2}\right) + 0.3\left[0.9\left(\frac{100}{2}\right) + 0.1\left(\frac{200}{2}\right)\right] = 35 + 0.3(45+10) = 35 + 16.5 = 51.5.$$

More Elaboration. Suppose the probability of revenue of 100 if one player choose High and the other chooses Low were z instead of 0.9. If z is too low, the equilibrium described above breaks down because Hardware finds it profitable to deviate to $High|Not\ Defective$.

$$\pi_{Hardware}(Low|Not\ Defective) = \frac{100}{2} = 50.$$

Deviating would yield Hardware a lower payoff:

$$\pi_{Hardware}(High|Not\ Defective) = z\left(\frac{100}{2}\right) + (1-z)\left(\frac{200}{2}\right) - 20 = 50z + 100 - 100z - 20.$$

This comes to be $\pi_{Hardware}(High|Not\ Defective) = 80 - 50z$, so if $z < 0.6$ then the payoff from $(High|Not\ Defective)$ is greater than 50, and so Hardware would be willing to unilaterally supply High effort even though Software is providing Low effort.

You might wonder whether Software would deviate from the equilibrium for some value of z even greater than 0.6. To see that he would not, note that

$$\pi_{Software}(High) = 0.7\left(\frac{100}{2}\right) + 0.3\left[z\left(\frac{100}{2}\right) + (1-z)\left(\frac{200}{2}\right)\right] - 20.$$

This takes its greatest value at $z = 0$, but even then the payoff from *High* is just $0.7(50) + 0.3(100) - 20 = 45$, less than the payoff of 50 from *Low*. The chances of non-defective parts are just too low for Software to want to take the risk of playing *High* when Hardware is sure to play *Low*.

This situation is like that of two people trying to lift a heavy object. Maybe it is simply too heavy to lift. Otherwise, if both try hard they can lift it, but if only one does, his effort is wasted.

(c) What is Software's belief, in equilibrium, as to the probability that Hardware chooses low effort?

Answer. One. In equilibrium, Hardware always chooses *Low*.

(d) If Software sees that revenue is 100, what probability does he assign to defective parts if he himself exerted high effort and he believes that Hardware chose low effort?

Answer. 0.72 $(= (1)\frac{0.7}{(1)(0.7)+(0.9)(0.3)})$.