PROBLEM SET 5 ANSWERS

2 November 2006.

Problems: 10.2, 10.4, 10.5.

I have to apologize about problem 10.4. I changed the question back this spring, and forgot to work out and change the answers. It should have said $1.2 < a_1 < a_2 < 7$, not $a_1 < a_2 < 1$ in the question, and the answers I had online were wrong.

10.2. Task Assignment
Table 1 shows the payoffs in the following game. Sally has been hired by Rayco to do either Job 1, Job 2, or to be a Manager. Rayco believes that Tasks 1 and 2 have equal probabilities of being the efficient ones for Sally to perform. Sally knows which task is efficient, but what she would like best is a job as Manager that gives her the freedom to choose rather than have the job designed for the task. The CEO of Rayco asks Sally which task is efficient. She can either reply "Task 1," "Task 2," or be silent. Her statement, if she makes one, is an example of "cheap talk," because it has no direct effect on anybody's payoff (see Joseph Farrell & Matthew Rabin, "Cheap Talk," *Journal of Economic Perspectives*, 10: 103-118 [Summer 1996]).

Table 1: The Right To Silence Game Payoffs

		Sally's Job		
		Job 1	Job 2	Manager
	Task 1 is efficient (.5)	2,5	$1,-2$	3, 3
Sally knows:				
	Task 2 is efficient (.5)	$1,-2$	2,5	3,3

Payoffs to: (Sally, Rayco).

(a) If Sally did not have the option of speaking, what would happen?

<u>Answer.</u> Rayco would make her a Manager. Rayco's payoff is 3 then, but a deviation to either Job 1 or Job 2 would yield a payoff of .5(5) + .5(-2) = 1.5. Sally has no choices to make.

(b) There exist perfect Bayesian equilibria in which it does not matter how Sally replies. Find one of these in which Sally speaks at least some of the time, and explain why it is an equilibrium. You may assume that Sally is not morally or otherwise bound to speak the truth.

<u>Answer.</u> The key to answering this question and part (c) is to know what a perfect bayesian equilibrium is: a strategy for each player, plus any out-of-equilibrium beliefs that are needed. Someone who remembers that a strategy must specify what Sally does in each of the two states of the world and what Rayco does in response to each of Sally's three possible actions is a long ways towards answering the questions correctly. Here, try the following equilibrium:

Sally: Always say "Task 1." Rayco: Give Sally the job as Manager, regardless of her message or whether she is silent. Out-of-equilibrium belief: Rayco thinks the probability that Task 1 is efficient is 0.5 if Sally says Task 2 or is silent.

Sally's payoff is 3, and she cannot change it by deviating. Rayco's payoff is 3, but a deviation to either Job 1 or Job 2 would yield a payoff of .5(5) + .5(-2) = 1.5.

This is an example of a "babbling equilibrium," so called because the uninformed player treats the informed player's cheap talk as meaningless babbling. Since it doesn't matter what Sally says, there are lots of babbling equilibria, in each of which she says something different–but always meaningless. Note that if the message were costly, though, that would reduce the number of equilibria.

(c) There exists a perverse variety of equilibrium in which Sally always tells the truth and never is silent. Find an example of this equilibrium, and explain why neither player would have incentive to deviate to out-of-equilibrium behavior.

Answer. Sally: Say Task 1 if Task 1 is efficient. Say Task 2 if Task 2 is efficient. Rayco: If Sally says Task 1, give her Job 1. If Sally says Task 2, give her Job 2. If Sally is silent, give her Job 1. Out-of-equilibrium belief: If Sally is silent, then Task 1 is efficient.

Sally will tell the truth because if she deviates and the wrong task is assigned, her payoff will be 1 instead of 2. In particular, if she deviates and is silent, she will be given Job 1. Rayco has no incentive to deviate, because given that Sally always tells the truth, Rayco's payoff would fall from 5 to -2 from a deviation. If Sally is silent, which never happens in equilibrium, then Rayco's belief requires that Rayco give her Job 1 in order to maximize Rayco's payoff.

This out-of-equilibrium belief is not particularly plausible, and Farrell and Rabin use this as an example of an implausible equilibrium. It is good for learning how to describe equilibria, though!

10.4. Incentive Compatibility and Price Discrimination

Two consumers have utility functions $u_1(x_1, y_1) = a_1 log(1 + x_1) + y_1$ and $u_2(x_2, y_2) = a_2 log(1 + x_2) + y_2$, where $2 < a_1 < a_2 < 7$. The price of the y-good is 1 and each consumer has an initial wealth of 15. A monopolist supplies the x-good. He has a constant marginal cost of 1.2 up to his capacity constraint of 10. He will offer at most two price-quantity packages, (r_1, x_1) and (r_2, x_2), where r_i is the total cost of purchasing x_i units. He cannot identify which consumer is which, but he can prevent resale.

(Note: in the book, this problem had the error $a_1 < a_2 < 1$, which would result in zero sales being optimal, instead of $2 < a_1 < a_2 < 7$.)

(a) Write down the monopolist's profit maximization problem. You should have four constraints plus the capacity constraint.

Answer. The utility functions are quasilinear– we can think of y_1 as "all other goods" or "money", since its price is one. When r_1 rises by 1 dollar, y_1 falls by 1 dollar. Or, put differently, Consumer 1's utility from good y will be $15 - r_1$. Thus, we can rewrite the utility function as $u_1(x_1) = a_1 log(1 + x_1) - r_1$, dropping the 15 since it is just a constant and won't affect the maximizing.

The profit maximization problem is then to maximize $r_1 + r_2 - 1.2(x_1 + x_2)$ subject to two participation constraints, two self selection constraints, and a capacity constraint:

$$
\begin{aligned}
a_1 log(1 + x_1) - r_1 &\geq 0 \\
a_2 log(1 + x_2) - r_2 &\geq 0 \\
a_1 log(1 + x_1) - r_1 &\geq a_1 log(x_2) - r_2 \\
a_2 log(1 + x_2) - r_2 &\geq a_2 log(1 + x_1) - r_1 \\
x_1 + x_2 &\leq 10
\end{aligned}
$$

(b) Which constraints will be binding at the optimal solution?

Answer. Unless the seller decides not to sell to the low-valuing Consumer 1 at all, Consumer 1's participation constraint will be binding, so $a_1 x_1 - r_1 = 0$. Otherwise, the seller could just raise the price to Consumer 1 a little and increase his payoff.

Consumer 2 values the good more, and hence will be made to pay the higher price for the larger number of units. The seller cannot take away all of Consumer 2's surplus, because Consumer 2 always has the option to pay the low price for the smaller number of units, but he can reduce Consumer 2's surplus to the point where Consumer 2 is indifferent between those alternatives.

Possibly, the capacity constraint will also be binding. We have to wait till part (c) to check on that, though.

Thus, the constraints that are always binding are $a_1 log(x_1) - r_1 = 0$ and $a_2 log(x_2) - r_2 = a_2 log(x_1) - r_1$.

If the seller decided to sell just to Consumer 2, then Consumer 2's participation constraint would be binding, and the seller does not worry about offering Consumer 1 a separate contract, so the other three constraints are inapplicable.

(c) Substitute the binding constraints into the objective function. What is the resulting expression? What are the first-order conditions for profit maximization? What are the profit-maximizing values of x_1 and x_2?

Answer. Profit is

$$r_1 + r_2 - 1.2(x_1 + x_2).$$

The binding constraints tell us that $r_1 = a_1 log(1+x_1)$ and $r_2 = a_2 log(1+x_2) - a_2 log(1+x_1) + r_1 = a_2 log(1+x_2) - a_2 log(1+x_1) + a_1 log(1+x_1)$. Subsituting for r_1 and r_2 from the binding constraints yields a profit of

$$a_1 log(1+x_1) + a_2 log(1+x_2) - a_2 log(1+x_1) + a_1 log(1+x_1) - 1.2(x_1 + x_2)$$

or

$$2a_1 log(1+x_1) + a_2 log(1+x_2) - a_2 log(1+x_1) - 1.2(x_1 + x_2).$$

Maximizing profit with respect to x_1 yields the first order condition

$$\frac{2a_1}{1+x_1} - \frac{a_2}{1+x_1} - 1.2 = 0,$$

so $2a_1 - a_2 = 1.2 + 1.2x_1$ and

$$x_1 = \frac{2a_1 - a_2}{1.2} - 1.$$

Maximizing profit with respect to x_2 yields the first order condition

$$\frac{a_2}{x_2 + 1} - 1.2 = 0,$$

so

$$x_2 = \frac{a_2}{1.2} - 1.$$

Note that this requires that $a_1 > 1.2$ and $a_2 > 1.2$. Otherwise, the marginal utility of the y-good is higher than the marginal utility of the x-good even at $x = 0$, so one or both consumers will not buy the x-good at all.

That is the answer if the capacity constraint is not binding. Is it? It says that $x_1 + x_2 \leq 10$, so we need

$$x_1 + x_2 = \frac{2a_1 - a_2}{1.2} - 1 + \frac{a_2}{1.2} - 1 = \frac{2a_1}{1}.2 - 2 \leq 10,$$

which requires $2a_1 \leq 1.2(12)$, that is, $a_1 \leq 7.2$.

If $a_1 > 7.2$, then the capacity constraint would be binding. If we know that $x_1 + x_2 = 10$ then $x_2 = 10 - x_1$ and we can rewrite the seller's profit maximization problem

$$\underset{x_1, x_2, r_1, r_2}{Maximize} \quad \{r_1 + r_2 - 2(x_1 + x_2)\}$$

which is subject to the constraints, as the unconstrained

$$\begin{aligned} \underset{x_1}{Maximize} \quad & a_1 log(1+x_1) + a_2 log(1+10-x_1) - a_2 log(1+x_1) + a_1 log(1+x_1) - 2(10) \\ & = 2a_1 log(1+x_1) + a_2 log(11-x_1) - a_2 log(1+x_1) - 20 \end{aligned}$$

which has first order condition

$$\frac{2a_1}{1+x_1} + \frac{(-1)a_2}{11-x_1} - \frac{a_2}{1+x_1} = 0.$$

That solves out as $(2a_1 - a_2)(11 - x_1) - a_2(1 + x_1) = 0$, so $22a_1 - 2a_1x_1 - 11a_2 + a_2x_1 - a_2 - a_2x_1 = 0$, so $22a_1 - 12a_2 = 2a_1x_1$ so

$$x_1 = 11 - \frac{6a_2}{a_1}.$$

(which makes $x_2 = 10 - (11 - \frac{6a_2}{a_1}) = \frac{6a_2}{a_1} - 1$)

This value of x_1 reaches its maximum at 5, if $x_2 = x_1$. If, however, $a_1 < \frac{6}{11}a_2$, x_1 is negative, which cannot happen. That is a corner solution. The seller does not find it worthwhile serving Consumer 1. Instead, $x_1 = 0$, $x_2 = 10$, and $r_2 = a_2 log(1 + x_2) = a_2 log(11)$.

10.5. The Groves Mechanism

A new computer costing 10 million dollars would benefit existing Divisions 1, 2, and 3 of a company with 100 divisions. Each divisional manager knows the benefit to his division (variables $v_i, i = 1, ..., 3$), but nobody else does, including the company CEO. Managers maximize the welfare of their own divisions. What dominant strategy mechanism might the CEO use to induce the managers to tell the truth when they report their valuations? Explain why this mechanism will induce truthful reporting, and denote the reports by $x_i, i = 1, ..., 3$. (You may assume that any budget transfers to and from the divisions in this mechanism are permanent– that the divisions will not get anything back later if the CEO collects more payments than he gives, for example.)

Answer. Let Division 1 pay $(10 - x_2 - x_3)$, Division 2 pay $(10 - x_1 - x_3)$, and Division 3 pay $(10 - x_1 - x_2)$ if the computer is bought, where that payment could be negative, and buy the computer if $x_1 + x_2 + x_3 \geq 10$.

Manager i's report does not affect its payment except by affecting whether the computer is bought. Let us take the case of Manager 1 for concreteness. His payoff is $v_1 - (10 - x_2 - x_3)$ if the computer is bought and 0 otherwise. He therefore wants the computer to be bought if and only if $v_1 + x_2 + x_3 \geq 10$. By reporting $x_1 = v_1$, he achieves exactly that outcome– the computer is bought only when he wants it to be bought. If the other two divisions overreport, he wants the computer to be bought because the mechanism will make him pay less than x_1, and if they underrport, he wants it not to be bought, because the mechanism will make him pay more than x_1.