PROBLEM SET 6 ANSWERS

16 November 2006.

Problems 11.1, 11.4, 11.6, 13.1.

11.1. Is Lower Ability Better?
Change Education I so that the two possible worker abilities are $a \in \{1, 4\}$.

(a) What are the equilibria of this game? What are the payoffs of the workers (and the payoffs averaged across workers) in each equilibrium?

Answer. The pooling equilibrium is

$$s_L = s_H = 0, w_0 = w_1 = 2.5, Pr(L|s = 1) = 0.5, \tag{1}$$

which uses passive conjectures: . The payoffs are $U_L = U_H = 2.5$, for an average payoff of 2.5.

The separating equilibrium is

$$s_L = 0, s_H = 1, w_0 = 1, w_1 = 4. \tag{2}$$

The payoffs are $U_L = 1$ and $U_H = 2$, for an average payoff of 1.5 . This equilibrium can be justified by the self selection constraints

$$U_L(s = 0) = 1 > U_L(s = 1) = 4 - 8/1 = -4 \tag{3}$$

and

$$U_H(s = 0) = 1 < U_H(s = 1) = 4 - 8/4 = 2. \tag{4}$$

Thus, the payoff averaged across workers is 1.5 (=.5[1] + .5[2]).

(b) Apply the Intuitive Criterion (see N6.2). Are the equilibria the same?

Answer. Yes. The intuitive criterion does not rule out the pooling equilibrium in the game with $a_h = 4$. There is no incentive for *either* type to deviate from $s = 0$ even if the deviation makes the employers think that the deviator is high-ability. The payoff to a persuasive high-ability deviator is only 2, compared the 2.5 that he can get in the pooling equilibrium.

(c) What happens to the equilibrium worker payoffs if the high ability is 5 instead of 4?

Answer. The pooling equilibrium is

$$s_L = s_H = 0, w_0 = w_1 =, Pr(L|s = 1) = 0.5, \tag{5}$$

which uses passive conjectures. The payoffs are $U_L = U_H = 3$, with an average payoff of 3.

The separating equilibrium is

$$s_L = 0, s_H = 1, w_0 = 1, w_1 = 5. \tag{6}$$

The payoffs are $U_L = 1$ and $U_H = 3.4$ with an average payoff of 2.2. The self-selection constraints are

$$U_H(s = 0) = 1 < U_H(s = 1) = 5 - \frac{8}{5} = 3.4 \tag{7}$$

and

$$U_L(s = 0) = 1 > U_L(s = 1) = 5 - \frac{8}{1} = -3. \tag{8}$$

(d) Apply the Intuitive Criterion to the new game. Are the equilibria the same?

Answer. No. The strategy of choosing $s = 1$ is dominated for the Lows, since its maximum payoff is -3, even if the employer is persuaded that he is High. So only the separating equilibrium survives.

(e) Could it be that a rise in the maximum ability reduces the average worker's payoff? Can it hurt all the workers?

Answer. Yes. Rising ability would reduce the average worker payoff if the shift was from a pooling equilibrium when $a_h = 4$ to a separating equilibrium when $a_h = 5$. Since the Intuitive Criterion rules out the pooling equilibrium when $a_h = 5$, it is plausible that the equilibrium is separating when $a_h = 5$. Since the pooling equilibrium is pareto-dominant when $a_h = 4$, it is plausible that it is the equilibrium played out. So the average payoff may well fall from 2.5 to 2.2 when the high ability rises from 4 to 5. This cannot make every player worse off, however; the high-ability workers see their payoffs rise from 2.5 to 3.4.

[Below are some additional notes that are in first draft, since they came up in a recent class– be on the lookout for errors]

What if the signal were continuous instead of having to equal either 1 or 4? Then, a lower signal could still induce separation. Would that remove the perverse result of higher ability reducing average payoffs? The basic problem is that any signalling that goes on can't increase output. All the signalling does is to reduce the average payoff. On the other hand, increasing ability does increase the average payoff, as a direct effect. So the question is whether the increase in signalling cost outweighs the increase in output. Let's see what happens here:

The high-ability self-selection constraint would be

$$U_H(s=0) = 1 \leq U_H(s=s^*) = 5 - \frac{8s^*}{5}, \tag{9}$$

which means we would need $s^* \leq 2.5$.

The low-ability self-selection constraint would be

$$U_L(s=0) = 1 \geq U_L(s=1) = 5 - \frac{8s^*}{1}, \tag{10}$$

so s^* must be at least 0.5. If it is, then the separating payoffs are 1 for the low-ability and $5 - \frac{8(0.5)}{5} =$ 4 1/5 for the high-ability, an average payoff of 2.6.

The Intuitive Criterion results in the minimum necessary signal being used. Thus, use of it would argue for a shift from an average payoff of 2.5 to one of 2.6 when ability increased from 4 to 5 if the equilibrium shifted from pooling to separating. But the Intuitive Criterion would say that the pooling equilibrium would break down even when ability was 4, if the signal can be as low as 0.5. That's because the high-ability worker could deviate to signalling $s = 0.5$ and get payoff $4 - \frac{8*0.5}{4} = 4 - 1 = 3$, which is better than the 2.5 from pooling. In fact, the value of s^* when abilities are 1 and 4 can be even lower: solving $U_L(s=0) = 1 \geq U_L(s=1) = 4 - \frac{8s^*}{1}$ yields $s^* = 3/8$. That gives an average payoff of $(3.625 + 1)/2 = 2.3125$. So applying the Intuitive Criterion, higher ability now helps. If we don't apply it, though, a shift from pooling to separating might well reduce average welfare, and could even reduce the welfare of both players.

11.4. Signalling with a Continuous Signal

Suppose that with equal probability a worker's ability is $a_L = 1$ or $a_H = 5$, and the worker chooses any amount of education $y \in [0, \infty)$. Let $U_{worker} = w - \frac{8y}{a}$ and $\pi_{employer} = a - w$.

(a) There is a continuum of pooling equilibria, with different levels of y^*, the amount of education necessary to obtain the high wage. What education levels, y^*, and wages, $w(y)$, are paid in the pooling equilibria, and what is a set of out-of-equilibrium beliefs that supports them? What are the incentive compatibility constraints?

Answer. A pooling equilibrium for any $y^* \in [0, 0.25]$ is

$$w = \left| \begin{array}{l} 1 \; if \; y \neq y^* \\ 3 \; if \; y = y^* \end{array} \right. \tag{11}$$

with the out-of-equilibrium belief that $Pr(L|(y \neq y^*)) = 1$, and with $y = y^*$ for both types.

The self selection constraints say that neither High nor Low workers want to deviate by acquiring other than y^* education. The most tempting deviation is to zero education, so the constraints are:

$$U_L(y^*) = w(y^*) - 8y^* \geq U_L(0) = w(y \neq y^*) \tag{12}$$

and

$$U_H(y^*) = w(y^*) - \frac{8y^*}{5} \geq U_H(0) = w(y \neq y^*). \tag{13}$$

The constraint on the Lows requires that $y^* \leq 0.25$ for a pooling equilibrium. Otherwise the Lows would deviate to $y = 0$.

(b) There is a continuum of separating equilibria, with different levels of y^*. What are the education levels and wages in the separating equilibria? Why are out-of-equilibrium beliefs needed, and what beliefs support the suggested equilibria? What are the self selection constraints for these equilibria?

Answer. A separating equilibrium for any $y^* \in [0.5, 2.5]$ is

$$w = \left| \begin{array}{l} 1 \; if \; y \neq y^* \\ 5 \; if \; y = y^* \end{array} \right. \tag{14}$$

with the out-of-equilibrium belief that $Pr(L|(y \neq y^*)) = 1$, and with $y_L = 0, y_H = y^*$. An out-of-equilibrium belief is needed because only $y = 0$ and $y = y^*$ occur in equilibrium, which leaves lots of other possibilities.

The self selection constraints say that High workers do not want to deviate by acquiring other than y^* of education (0 is most tempting), and the Lows do not want to deviate by acquiring y^* of education.

$$U_L(y^*) = w(y^*) - 8y^* \leq U_L(0) = w(y \neq y^*) \tag{15}$$

and

$$U_H(y^*) = w(y^*) - 8y^*/5 \geq U_H(0) = w(y \neq y^*). \tag{16}$$

These two constraints tell us that $y^* \geq 0.5$ and $y^* \leq 2.5$ in a separating equilibrium.

(c) If you were forced to predict one equilibrium which will be the one played out, which would it be?

Answer. The out-of-equilibrium beliefs are unsatisfactory in the pooling equilibria because acquiring more than $y = 0.5$ in education is a dominated strategy for the Low type. If one carries this reasoning further, only $y = 0.5$ is satisfactory, because separating equilibria with more signalling require the belief that $y = 0.5$ is a sign of a Low type.

11.6. Game Theory Books

In the Preface I explain why I listed competing game theory books by saying, "only an author quite confident that his book compares well with possible substitutes would do such a thing, and you will be even more certain that your decision to buy this book was a good one."

(a) What is the effect on the value of the signal if there is a possibility that I am an egotist who overvalues his own book?

Answer. The most direct effect is that this reduces the value of the signal, since I would then be underestimating the number of readers who would compare the two books and choose mine.

On the other hand, suppose it is common knowledge that I might overvalue my book. My willingness to list competitors would show even more strongly that my own opinion of my book is very high, since I do it knowing that with some probability it will be very costly to me for readers to compare.

(b) Is there a possible non-strategic reason why I would list competing game theory books?

Answer. Yes. It increases the value of my book, by adding useful information. This is the actual reason I list them– the signalling explanation is just a humorous aside.

(c) If all readers were convinced by the signal of providing the list and so did not bother to even look at the substitute books, then the list would not be costly even to the author of a bad book, and the signal would fail. How is this paradox to be resolved? Give a verbal explanation.

<u>*Answer*</u>. This has to be a mixed-strategy equilibrium. The reader will look at the list with some probability (and otherwise just buy the book), and the author of a bad book will provide the list with some probability.

The author of a good book will always provide the list. He doesn't care whether the reader looks at it or not.

Or, if the probability the book is good is high enough, a resolution to the paradox is that the signal indeed does fail, in the sense of not distinguishing good from bad books, but is still used. We could have a pooling equilibrium in which all authors provide the list, and the reader never checks.

(d) Provide a formal model for part (c).

<u>*Answer*</u>. Here is one possibility.

(0) Nature choose the book to be good with probability θ and otherwise bad.

(1) Nature chooses fraction γ of "passive" readers to buy the book regardless of quality (maybe they don't care about quality and just buy the first book they see).

(2) The author chooses whether to include a list of other books.

(3) The reader decides whether to look at the books on the list at cost c or not.

(4) The reader decides what book to buy, or whether not to buy at all.

The author has payoff b if the reader buys, whether his book is good or bad, and no direct cost of listing other books. The nonpassive workers have payoff x from a good book and 0 from a bad book.

13.1. Rent-Seeking

Two risk-neutral neighbors in 16th century England, Smith and Jones, have gone to court and are considering bribing a judge. Each of them makes a gift, and the one whose gift is the largest is awarded property worth £2,000. If both bribe the same amount, the chances are 50 percent for each of them to win the lawsuit. Gifts must be either £0, £900, or £2, 000.

(a) What is the unique pure-strategy equilibrium for this game?

<u>*Answer.*</u> Each bids £900, for expected profits of 100 each (=-900 + 0.5(2000)). This is an all-pay auction, but with restrictions on the bid amounts. Table A13.1 shows the payoffs (but also includes the payoffs for when the strategy of a bid of 1,500 is allowed). A player who deviates to 0 has a payoff of 0; a player who deviates to 2,000 has a payoff of 0. (0,0) is not an equilibrium, because the expected payoff is 1,000, but a player who deviated to 900 would have a payoff of 1,100.

Table A13.1: Bribes I

		Jones £0	Jones £900	Jones £1500	Jones £2000
	£0	1000,1000	0,1100	0,500	0,0
Smith:	£900	1100,0	100,100	-900, 500	- 900,0
	£1500	500,0	500, −900	−500, −500	−1500, 0
	£2000	0,0	0, −900	0, −1500	−1000, −1000

Payoffs to: (Smith, Jones).

(b) Suppose that it is also possible to give a £1500 gift. Why does there no longer exist a pure-strategy equilibrium?

<u>*Answer.*</u> If one player bids 0 or 900, the other would bid 1500, so we know 0 and 900 would not be used in equilibrium. If both player bid 1500, payoffs would be negative (= 2000/2-1500 each), so one could deviate to 0 and increase his payoff. If both bid 2000, one can profit by deviating to 0. If one

player bids 1500 and the other bids 2000, the one with the bid of 1500 loses, for a payoff of -1500, and would be better off deviating to 0. This exhausts all the possibilities.

(c) What is the symmetric mixed-strategy equilibrium for the expanded game? What is the judge's expected payoff?

Answer. Let (θ_0, θ_{900}, $\theta_{1,500}$,$\theta_{2,000}$) be the probabilities. It is pointless ever to bid 2,000, because it can only yield zero or negative profits, so $\theta_{2,000} = 0$. In a symmetric mixed-strategy equilibrium, the return to the pure strategies is equal and the probabilities add up to one, so

$$\begin{aligned} &\pi_{Smith}(0) = \pi_{Smith}(900) = \pi_{Smith}(1500) \\ &0.5\theta_0(2000) = -900 + \theta_0(2000) + 0.5\theta_{900}(2000) \\ &= -1500 + \theta_0(2000) + \theta_{900}(2000) + 0.5\theta_{1500}(2000), \end{aligned} \tag{17}$$

and

$$\theta_0 + \theta_{900} + \theta_{1500} = 1. \tag{18}$$

Solving out these three equations for three unknowns, the equilibrium is $(0.4, 0.5, 0.1, 0.0)$.

The judge's expected payoff is 1,200 $(= 2(0.5(900) + 0.1(1500)) = 2[450 + 150]))$.

Note: The results are sensitive to the bids allowed. Can you speculate as to what might happen if the strategy space were the whole continuum from 0 to 2000?

(d) In the expanded game, if the losing litigant gets back his gift, what are the two equilibria? Would the judge prefer this rule?

Answer. Table A13.2 shows the new outcome matrix. There are three equilibria: $x_1 = (900, 900), x_2 = 1500, 1500)$, and $x_3 = (2000, 2000)$.

Table A13.2: Bribes II

		Jones			
		£0	£900	£1500	£2000
	£0	1000,1000	0,1100	0,500	0,0
Smith:	£900	1100,0	**250,250**	0, 500	0,0
	£1500	500,0	500, 0	**250, 250**	0, 0
	£2000	0,0	0, 0	0, 0	**0, 0**

Payoffs to: (Smith, Jones).

The judge's payoff was 1200 under the unique mixed-strategy equilibrium in the original game. Now, his payoff is either 900, 1500, or 2000. Thus, whether he prefers the new rules depends on which equilibrium is played out in it.