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Cedars Math
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Proof That the Square Root of Two Is Irrational

Definition 1: A **prime number** is a whole number greater than 1 that is evenly divisible only by itself and 1. *Examples:* 2, 3, and 5, but not 1,4, and 6.

Definition 2: A number's **factors** are the whole numbers by which it is evenly divisible. *Example:* The factors of 12 are 1, 12, 2, 6, 3, and 4.

Definition 3: The **prime factors** of a number are the factors that are prime numbers. *Example:* The prime factors of 12 are 2 and 3.

Definition 4: The **prime factorization** of a number is how it is formed by multiplying its prime factors. *Example:* The prime factorization of 12 is $2 \cdot 2 \cdot 3$, or equivalently, $2^2 \cdot 3$.

Definition 5: A whole number is **even** if one of its prime factors is 2. If it is not even it is **odd**. *Example:* Twelve is even because its prime factors are 2 and 3. Fifteen is odd because its prime factors are 3 and 5.

Definition 6: A **lemma** is a claim that is true but is interesting mainly for proving some other claim.

Definition 7: A **rational** number is a number that can be written as a fraction. If a number is not rational, it is **irrational.** *Example:* 3.5 is a rational number because it can be written as $7/2$; π is irrational because it cannot be written as a fraction a/b for any whole numbers a and b.

Definition 8: Two mathematical statements are **equivalent** if they say the same thing. *Example:* The ratio 2:1 is equivalent to the ratio 2 to 1.

Lemma 1a: If X is even, so is x^2. Only if x is even is x^2 also even.

Lemma 1b: If and only if x is even, so is x^2.

Lemmas 1 and 2 are equivalent.

Proof: $x^2 = x \cdot x$.
If x has the prime factors $m, n, \ldots q$, then $x \cdot x$ will have the same prime factors, just with double the exponents in the prime factorization.
(i) If x is even, 2 is one of its prime factors.
So x^2 will still have 2 as a prime factor, so it is even also.
(ii) If x isn't even, it's odd.
If x is odd, it does NOT have 2 as a prime factor.
Thus, x^2, with the same prime factors as x, won't have 2 as a prime factor.
(iii) So if x is even, x^2 is also, but if x isn't even, x^2 isn't either.
Q.E.D.

Theorem 1: The square root of 2 is irrational.

Proof. Suppose not.
Then there is some ratio a/b such that the fraction $a/b = \sqrt{2}$, where a/b is the fraction reduced as far as possible.
In that case, $a^2/b^2 = 2$, since that just squares both sides.
Then, $a^2 = 2b^2$, since that just multiplies both sides by b^2.
So a^2 is even.
By Lemma 1, since a^2 is even, so is a, since only an even number can be squared to get an even number.
That means we can write $a = 2k$, for some number k.
Then, $a^2 = (2k)^2 = 2 \cdot k \cdot 2 \cdot k = 4k^2$.
Since $a^2 = 2b^2$, that means $4k^2 = 2b^2$.
Then dividing both sides by two gives us $2k^2 = b^2$.
So b^2 is even.
By Lemma 1, since b^2 is even, so is b, just like a was.
But then a/b isn't the reduced fraction, since it has a factor of 2 on both top and bottom.
So it can't be that $\sqrt{2} = a/b$, reduced, for whole numbers a and b.
So $\sqrt{2}$ is irrational.
Q.E.D.