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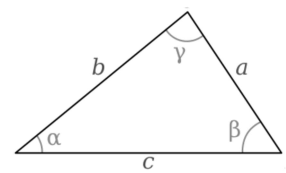
Heron's formula

In geometry, **Heron's formula** (or **Hero's formula**) gives the area of a triangle in terms of the three side lengths a, b, c. If $s = \tfrac{1}{2}(a + b + c)$ is the semiperimeter of the triangle, the area A is,[1]

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$



A triangle with sides *a*, *b*, and *c*

It is named after first-century engineer Heron of Alexandria (or Hero) who proved it in his work *Metrica*, though it was probably known centuries earlier.

Example

Let $\triangle ABC$ be the triangle with sides $a = 4$, $b = 13$ and $c = 15$. This triangle's semiperimeter is

$$s = \frac{a+b+c}{2} = \frac{4+13+15}{2} = 16$$

and so the area is

$$\begin{aligned} A &= \sqrt{s\left(s-a\right)\left(s-b\right)\left(s-c\right)} = \sqrt{16 \cdot (16-4) \cdot (16-13) \cdot (16-15)} \\ &= \sqrt{16 \cdot 12 \cdot 3 \cdot 1} = \sqrt{576} = 24. \end{aligned}$$

In this example, the side lengths and area are integers, making it a Heronian triangle. However, Heron's formula works equally well in cases where one or more of the side lengths are not integers.

Alternate expressions

Heron's formula can also be written in terms of just the side lengths instead of using the semiperimeter, in several ways,

$$\begin{aligned}A &= \tfrac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}\\ &= \tfrac{1}{4}\sqrt{2(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)}\\ &= \tfrac{1}{4}\sqrt{(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)}\\ &= \tfrac{1}{4}\sqrt{4(a^2b^2+a^2c^2+b^2c^2)-(a^2+b^2+c^2)^2}\\ &= \tfrac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}.\end{aligned}$$

After expansion, the expression under the square root is a quadratic polynomial of the squared side lengths a^2, b^2, c^2.

The same relation can be expressed using the Cayley–Menger determinant,

$$-16A^2 = \begin{vmatrix} 0 & a^2 & b^2 & 1 \\ a^2 & 0 & c^2 & 1 \\ b^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

History

The formula is credited to Heron (or Hero) of Alexandria (fl. 60 AD),[2] and a proof can be found in his book *Metrica*. Mathematical historian Thomas Heath suggested that Archimedes knew the formula over two centuries earlier,[3] and since *Metrica* is a collection of the mathematical knowledge available in the ancient world, it is possible that the formula predates the reference given in that work.[4]

A formula equivalent to Heron's, namely,

$$A = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2}$$

was discovered by the Chinese. It was published in *Mathematical Treatise in Nine Sections* (Qin Jiushao, 1247).[5]

Proofs

There are many ways to prove Heron's formula, for example using trigonometry as below, or the incenter and one excircle of the triangle,[6] or as a special case of De Gua's theorem (for the particular case of acute triangles),[7] or as a special case of Brahmagupta's formula (for the case of a degenerate cyclic quadrilateral).

Trigonometric proof using the law of cosines

A modern proof, which uses algebra and is quite different from the one provided by Heron, follows.[8] Let a, b, c be the sides of the triangle and α, β, γ the angles opposite those sides. Applying the law of cosines we get