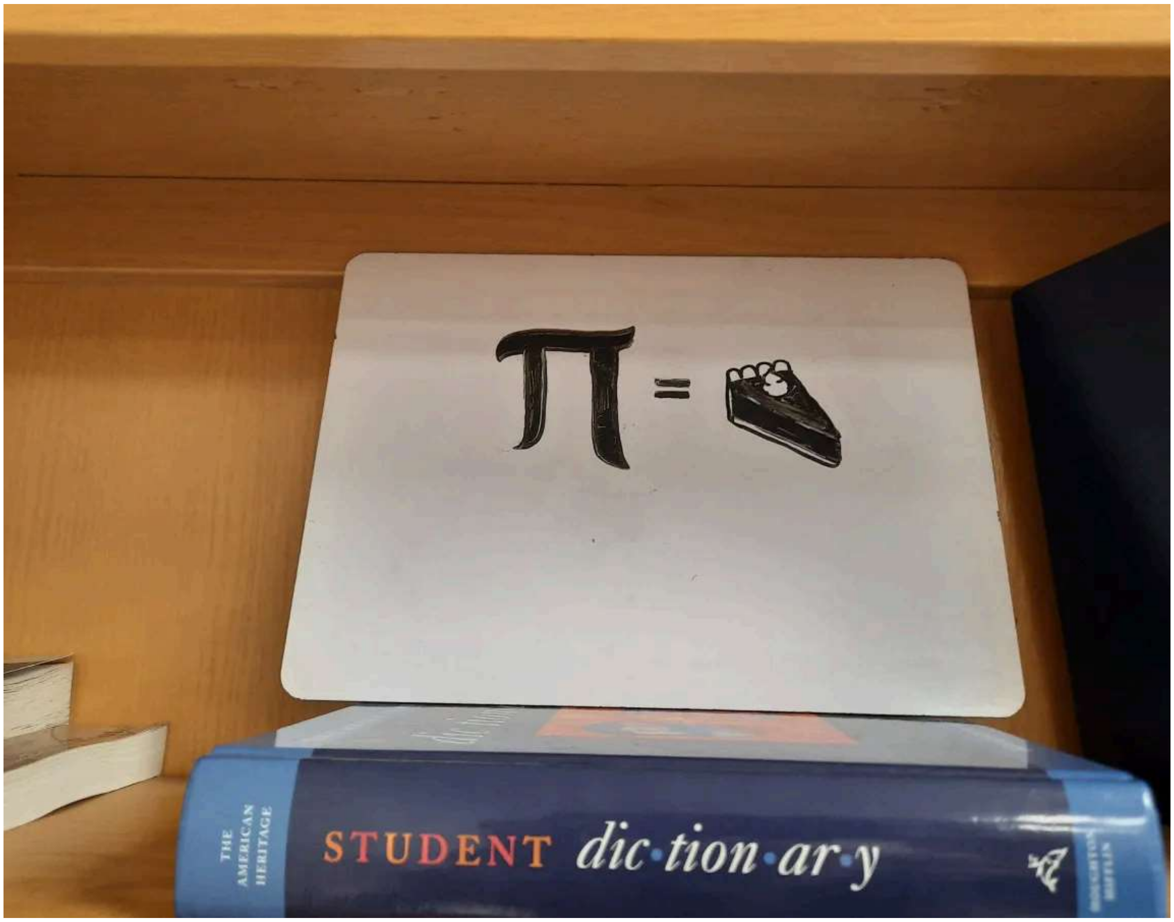
Pi Day Friday

Pi, part I

ERIC RASMUSEN
MAR 8, 2024

Share



Credit: Olivet, who drew the pi-pie

Earlier this week I wrote about how to explain that (-2) times (-3) equals 6, not -6. In one of my footnotes, I said, "Maybe I should Substack on Pi [1] separately someday." Reader Dr. Joe Horton commented, "How about a week from tomorrow? It's Pi Day." [2] Like myself (PhD, Course XIV, '84), Joe is an MIT alum (Course V, '69), and celebrating Pi Day is an old MIT tradition. [3] Pi Day is actually next Thursday, but as with Christmas, if you wait till the big day itself you miss half the fun. You need to prepare. Some of you, of course, have been doing just that ever since the excitement of President's Day died down, but others of you have been procrastinating and need a gentle reminder. So here it is. It's Pi Day Preparation Friday.

Let the festivities begin!

The first thing you need to know is that Pi is not

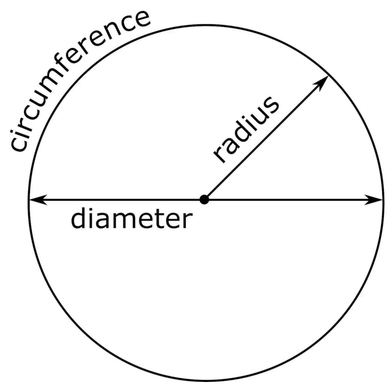
3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982148086513282306647093844609550582231725359408128481117450284102701938521105559644622948954930381964428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273724587006606315588174881520920962829254091715364367892590360011330530548820466521384146951941511609433057270365759591953092186117381932611793105118548074462379962749567351885752724891227938183011949 12

That's not even close. It's just an estimate. It's a bad estimate. A much better estimate is Pi = 3. The usual estimate is 3.14, or, if you want something more euphonious but less useful, 3.14159.

Let me explain. There are lots of ways to define Pi mathematically, all of them obtaining the exact same value. The oldest and best one is:

$$\pi = \frac{C}{d},$$

where C is the circumference and d is the diameter of a circle



But C/d is an irrational number. That mean if you try to write it out, it has an infinite number of digits. You could keep writing forever and not come to the end, and no pattern within the digits ever repeats itself an infinite number of times, so it isn't infinite just in the sense that 1/3 = 0.333333333 . . . or 1/7 = 0.142857 142857 14 . . . is infinite. [4]

Pi being irrational has an interesting implication. Suppose we make a code where 01: = A, 02: = B, . . . 26: = Z. Using that code "Pi Day" would be 1609040125. Since Pi has an infinite number of digits with no pattern repeating infinitely, with probability 1 we will eventually see 1609040125 1609040125 1609040125 . . . with 100 repetitions, the Pi Day segment of Pi. In fact, eventually somewhere in the list of digits of Pi the code for the entire Bible will appear. Kabbalists, take note. [5]

At any rate, the long estimate above is still infinitely far from being a complete list of the digits of Pi. I say it is a bad estimate because it is useless. It is too short to be accurate and too long to be useful. The estimate 3.14 is much better, because it's easy to memorize, pretty accurate, and crucially important to avoiding the fiasco of forgetting to buy a Pi Day present for your wife.

A better estimate of Pi than either the long one or 3.14 is simply to use Pi = 3. That's pretty close, and its easier to remember. As we'll see below, if you forget the number 3 you can pretty easily prove that Pi must be somewhere around 3. Sensible rounding like that also is the answer to the scoffer's claim that the Bible is bunk because it says of Hiram, King of Tyre and friend of King Solomon,

> And he made a molten sea, ten cubits from the one rim to the other it was round all about, and. . . a line of thirty cubits did compass it round about.... And it was an hand breadth thick. . . ." — I Kings 7: 23, 26.

Imagine the alternative:

> And he made a molten sea, ten cubits from the one rim to the other it was round all about, and. . . a line of thirty-one point four one give nine two six five three five eight nine seven nine three two three eight four six and a bit more cubits did compass it round about.... And it was an hand breadth thick. . . ." — I Kings 7: 23, 26.

God is not a pedant.

So 3 is a pretty good estimate, and so is 3.14. An unhappy compromise would be to take Pi to one decimal, to 3.1. I don't think that's ever done, but, shamefully, my own state of Indiana almost defined Pi as 3.2 as a matter of law. In 1897, a bill to that effect passed the House unanimously, after being introduced by the Representative from Evansville, one of whose constituents thought he could "square the circle" if only Pi equalled 3.2. [6] Fortunately a Purdue math professor happened to be in town to lobby for the university budget, and after he turned his lobbying skills onto some senators the bill died.

Sometimes you do need more digits. NASA uses 15 digits. That's enough to get orbits within half an inch of their true value: [7]

> By cutting pi off at the 15th decimal point, we would calculate a circumference for that circle that is very slightly off. It turns out that our calculated circumference of the 30-billion-mile (48-billion-kilometer) diameter circle would be wrong by less than half an inch (about one centimeter). Think about that. We have a circle more than 94 billion miles (more than 150 billion kilometers) around, and our calculation of that distance would be off by no more than the width of your little finger. (The Jet Propulsion Lab's Chief Engineer for Mission Operations and Science, Marc Rayman in "How Many Decimals of Pi Do We Really Need?" (2022))

I don't think anyone really needs more digits of Pi than NASA does, but the article goes on to talk about what you can do with 38 digits:

> The radius of the universe is about 46 billion light years. Now let me ask (and answer!) a different question: How many digits of pi would we need to calculate the circumference of a circle with a radius of 46 billion light years to an accuracy equal to the diameter of a hydrogen atom, the simplest atom? It turns out that 37 decimal places (38 digits, including the number 3 to the left of the decimal point) would be quite sufficient.

Very often, people suggest 22/7 as a reasonable compromise between precision and simplicity. The value 22/7, which equals 3 1/7, indeed is simple, with only 4 symbols needed, the same as 3.14. The problem is that it's not useful, unless you plug it into a calculator and decimalize it, e.g. Pi = 22/7 = 3.1428571428. Try measuring 1/7 of something by sight. Or even use a ruler, when you measure 1/7 of something 4.56 feet long, or 3'6 3/8". It's a lot harder than 1/6, and much much harder than 1/8. That's why you'll never see a pizza with seven slices. If you're going to go down the fraction route, use Pi = 3 1/8, which equals 3.125. If you want more precision, bump up your answer a mite, so it's closer to 3.14.

But how do we come up with these values at all, even the value Pi = 3? You can eyeball it, but it's hard to compare the length of a line (the diameter) with a curve (the circumference). There's actually a way to prove it's between 2 and 4 that I showed my 7th grade class, and it's only a little bit harder to show it's between $2\sqrt{2}$ (which is about 2.8) and 4. I've saved that for the end of this Substack, though, since even easy proofs will glaze many readers' eyes. What I like to tell my students, though, is that there's often both a scientific and a mathematical approach to finding the size of something. The scientific approach is to just measure it. So that's their homework: measure Pi. I tell them to find two different circles at home, and measure the diameter using and circumference of each using a ruler and a string, and then compute Pi_i = Circumference/diameter. Do this five times for each, so i = 1,2,3,4,5, compute the average for each circle, and compute the grand average for all your measurements. Then I take the grand average of the class by averaging all 5 students (I have a small class this year). Here are the results: (Asher only did i = 1,2,3 for some reason)

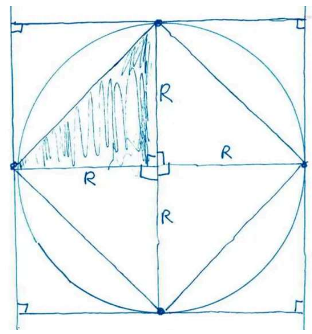
	A	B	C	D	E	F
1		Asher	Emma	Zoe	Lyndon	Jon
2	First	2.94	2.80	3.57	3.09	3.03
3		2.83	2.89	3.07	3.20	2.70
4	Circle	3.00	2.84	3.16	3.19	2.71
5			3.17	3.22	3.22	2.98
6			2.90	3.10	3.16	2.97
7	AVERAGE	**2.93**	**2.92**	**3.22**	**3.17**	**2.88**
8						
9		2.81	2.60	3.08	3.24	3.00
10	Second	3.06	2.94	2.99	3.21	3.29
11		2.81	3.00	3.13	3.26	3.33
12	Circle		2.80	3.24	3.24	2.75
13			2.90	3.29	3.24	3.14
14	AVERAGE	2.90	2.85	3.18	3.24	3.10
15						
16	Average	**2.91**	**2.88**	**3.20**	**3.21**	**2.99**
17						
18	Class Average	**3.04**				
19						

So our scientific estimate is Pi = 3.04. This seems to be pretty similar across the ten circles, so we could conclude that Pi is the same for all circles, which is sort of true. [8]

We can also estimate the value of Pi mathematically. The scientific method is inductive. We use a combination of definitions and measurements to come up with an answer. It is not certain. There is measurement error. There might be massive measurement error. And then we used some math on it too. We took averages. We might make a mistake taking the average. I actually did, the first time I made up the spreadsheet. I put the wrong cell range into one of the average formulas.

The mathematical approach is purer. It is uses deduction. We start with sure premises, and reason logically from them. I won't go all Euclidy on you and start with postulates, but let's see what can be simply deduced.

Let's go back to our start. The definition of Pi is the circumference of a circle divided by its diameter. When we say Circumference = Pi*diameter, what we are really saying is that whatever the size of the circle, the Circumference/Diameter is the same number, and we call that number Pi. How big Pi is? You could just measure itusing a string to find out the length of the circumference and then use a ruler to find out the diameter. That's what we just did. Since we know all circles have the same Circumference/Diameter, that would be a good estimate. We can also try to prove it mathematically, instead of scientifically, using logic instead of measurement. That's what we'll do next using this diagram, which I'll copy several time at convenient places. I do this in terms of the radius, where Diameter = 2*Radius, because otherwise we would get a lot of 1/2 fractions in our reasoning.



Theorem 1: *Pi is less than 4.*

Proof: We can draw a square to enclose the circle, touching it at four points on west, east, north and south. Each side of the square will have length 2R

Thus, the perimeter of the square is 4*2R, which equals 8R, which equals 4*Diameter.

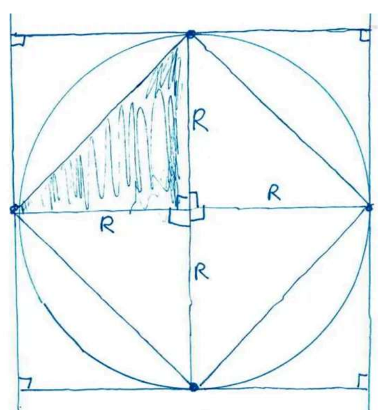
The perimeter is greater than the circumference, so

4*Diameter > Circumference, which means 4 > Circumference/Diameter = Pi,

so

Pi = Circumference/Diameter < 4.

Q.E.D.



Theorem 2: *Pi is greater than 2.*

Proof: We can draw a square just inside the circle, touching it at four points on west, east, north and south, though this square will be tilted. The area of the circle is Pi*R*R.

The area of the square can be found by dividing it into four equal-sized triangles.

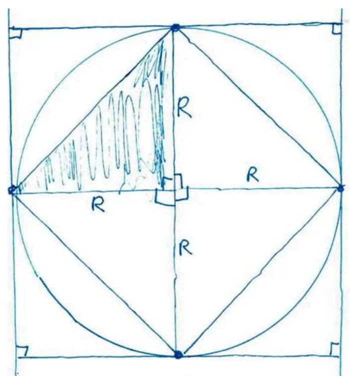
Each triangle has a base of R and a height of R, so it has area R*R/2.

Adding them all up, we get the area of the square to be 4*(R*R/2), which equals 2*R*R.

The area of the circle is greater than the area of the square, since the square is inside the circle, so

Pi*R*R > 2*R*R,

so Pi > 2. Q. E. D.



Theorem 3 will get our lower and upper bound interval to be tighter than Theorem 1, where it was (2,4). We'll need to use the Pythagorean Theorem, though. [9]

Theorem 3: *Pi is greater than 2 sqrt (2).*

Proof: We can draw a square just inside the circle, touching it at four points on west, east, north and south, though this square will be tilted. The perimeter of the circle is Pi*(2R).

The perimeter of the square is four times the long side of the shade triangle. We can find that long side using the Pythagorean Theorem: Long^2 = Short_1^2 + Short_2^2, so here, Long^2 = R^2 + R^2 so Long^2 = 2R^2, so Long = sqrt (2R^2) = sqrt(2)*R. The perimeter is four times that, so it is 4*Sqrt(2)*R.

We know that the perimeter of the square is less than the circumference of the circle, so

4 \sqrt{2} R < 2 R Pi

Thus, 2 \sqrt{2} < Pi

Q. E. D.

This kind of inside-square outside-square method can be used to get tighter and tighter bounds for Pi. I think this is the way Archimedes did it. I told my church friend Professor Christopher Connell about what I was doing with the 7th graders, and he responded with a long email on using rounder and rounder polygons to estimate the value of Pi using a little trigonometry. Here's what he said, somewhat rewritten by me:

> To prove Pi is larger than 3 one can proceed as follows. If you have a sector of angle Theta, then the length of the linear crossmember intersecting the circle at the two endpoints of the sector is 2 Sin(Theta/2). The perimeter of an inscribed n-gon is then
>
> Perimeter = 2 n Sin(Pi/n)
>
> This is always strictly less than 2 Pi since the line segments are geodesic and hence shorter than the circular arcs.
>
> So to prove Pi is bigger than 3, we want to find the first n such that n Sin(Pi/n) >= 3. This happens at n = 6. When n = 6, in fact, the value is exactly 3.
>
> Here are the first 12 polygons. [10]

n	$n \sin\left(\frac{\pi}{n}\right)$
1	0
2	2
3	$\frac{3\sqrt{3}}{2}$
4	$2\sqrt{2}$
5	$5\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}$
6	3
7	$7\sin\left(\frac{\pi}{7}\right)$
8	$8\sin\left(\frac{\pi}{8}\right)$
9	$9\sin\left(\frac{\pi}{9}\right)$
10	$\frac{5}{2}(\sqrt{5}-1)$
11	$11\sin\left(\frac{\pi}{11}\right)$
12	$3\sqrt{2}\,(\sqrt{3}-1)$

Here are the values we get from the first 100 regular n-gons inscribed within the circle:

{0, 2, 2.59808, 2.82843, 2.93893, 3, 3.03719, 3.06147, 3.07818, 3.09017, 3.09906,
3.10583, 3.1111, 3.11529, 3.11868, 3.12145, 3.12374, 3.12567, 3.1273,
3.12869, 3.12989, 3.13093, 3.13183, 3.13263, 3.13333, 3.13395, 3.13451,
3.13501, 3.13545, 3.13585, 3.13622, 3.13655, 3.13685, 3.13712, 3.13738,
3.13761, 3.13782, 3.13802, 3.1382, 3.13836, 3.13852, 3.13866, 3.1388,
3.13892, 3.13904, 3.13915, 3.13925, 3.13935, 3.13944, 3.13953, 3.13961,
3.13968, 3.13975, 3.13982, 3.13988, 3.13995, 3.14, 3.14006, 3.14011, 3.14016,
3.1402, 3.14025, 3.14029, 3.14033, 3.14037, 3.14041, 3.14044, 3.14048,
3.14051, 3.14054, 3.14057, 3.1406, 3.14062, 3.14065, 3.14067, 3.1407,
3.14072, 3.14074, 3.14076, 3.14079, 3.14081, 3.14082, 3.14084, 3.14086,
3.14088, 3.14089, 3.14091, 3.14093, 3.14094, 3.14095, 3.14097, 3.14098,
3.141, 3.14101, 3.14102, 3.14103, 3.14104, 3.14105, 3.14107, 3.14108}

I'd intended to spend most of this Substack talking about using computer estimate of Pi and about other formulas for Pi besides Pi = Circumference/Diameter, but it's gotten very long already and I have a lot of free-speech agitation to do today. So I'll continue another day, after leaving you with one more story from class.

One of the amazing things about Pi is how many formulas describe it exactly. You could take any one of them to be the definition, since they all generate the exact same number. To convey this idea, I gave my 7th graders three pages of 25 formulas for Pi that are relatively simple, where the word "relatively" does a lot of work. In fact, the approximate meaning of "relatively simple" here is "simple enough that Mr. Rasmusen doesn't have to ask Mr. Connell what the formula means." I asked them each to pick a formula for me to try to explain to them. All of the students picked one of these two:

$$\pi = \sum_{k=1}^{\infty} k \frac{2^k k!^2}{(2k)!} - 3 \qquad \text{(Plouffe's series)}$$

$$\pi = \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx\right)^2 \text{ (from the Normal density)}$$

I won't try to explain Plouffe's series and the Normal density to you now, but I hope this whets your appetite for Part II, whenever I get to it.

---

1 Why, you might ask, do I denote the number as Pi rather than pi or π? Good question. Usually I would use π, and I will use that sometimes in this substack, but readers deficient in both mathematics and classics would find it more difficult to read. Since π is lower case, pi would seem the next-best choice. But the word pi looks like an incomplete pin, pit, or pig. We need some contract, and I don't want to write "pi" or pi every time— that is too obtrusive. Capitalizing with the result Pi nicely separates and specializes the word, even though the upper-case letter in Greek is quite different—Π — which in mathematics is used for "Product", so Π (i=1..3) 1/i = (1/1) (1/2) (1/3) = 1/6.

2 I hope you didn't miss celebrating Pi Month (March 2014, 3.14) or Super Pi Day Minute (9:01p.m., March 14, 2015, 3.14159, since math people aren't awake at the crack of dawn). Of course, we all missed Pi Year (314 A.D.), the year of the First Council of Arles, which excommunicated conscientious objectors, no doubt by coincidence.

3 From Wikipedia:

> The Massachusetts Institute of Technology has often mailed its application decision letters to prospective students for delivery on Pi Day. Starting in 2012, MIT has announced it will post those decisions (privately) online on Pi Day at exactly 6:28 pm, which they have called "Tau Time", to honor the rival numbers pi and tau equally. In 2015, the regular decisions were put online at 9:26 am, following that year's "pi minute", and in 2020, regular decisions were released at 1:59 pm, making the first six digits of pi.

4 If you try to be clever and use use Base 2 or Base 7 instead of Base 10 numbers, Pi will still have infinite non-repeating digits. The only way that trick will work is if you use Base-Pi, in which case Pi = 10,

Pi**2 = 100, and so forth. But that will mess up all your other numbers, in the same way as defining Pi: = 3 will.

5 I haven't the heart to set you off to spending your entire life on a wild goose chase. The link between Pi and the Bible is intriguing, but the link also exists between any irrational number and any book, sacred or profane. For example, the square root of two is irrational. Its digits start like this:

$\sqrt{2} \approx 1.41421356237309504880168872420969807856967187537694$

At the website Pisearch.org you can search for any pattern you like within Pi, E, or $\sqrt{2}$. For example, the string 66666 first appears at the 48,439th decimal digit of Pi, but it doesn't appear until the 226,697th decimal digit of $\sqrt{2}$.

There are an infinite number of irrational numbers, and, in a weird but useful sense, there are infinitely more of them than there are of rational numbers, so many more that if you picked a number from 1 to 10 randomly, the probability you picked a rational number would be zero.

6 The constituent was wrong, I think. True, the basic reason you can't square the circle is because you can't construct transcendental numbers like Pi using the axioms of Euclid (which are equivalent to having a compass and straightedge), unlike simpler irrational numbers like $\sqrt{2}$ that come from simple operations like square rooting. But think if you define Pi := 3.2 then transcendentalism must pop up somewhere else in the construction.

7 I almost wrote that NASA uses 15 digits of Pi and gets obits within 1 foot of the true value, but, alas, obits are much harder to get right.

8 "Pi is the same for all circles, which is sort of true." I am sure some readers are saying "What! It's not just sort of true, it's ALWAYS true, if you could measure exactly and it is an exact circle." Well, no. You're forgetting about non-Euclidean spaces. "Well, that's being picky," you will say. "That's extremely abstruse, and you can't expect your 7th graders to think about that." But I can! Continue reading this Substack.

9 To be sure, we used the formula Area = Pi*R*R in Theorem 2, and that formula is actually much more advanced mathematically than the Pythagorean Theorem, but A = Pi*R*R is easier to understand and memorize so students learn it first.

10 We can only use the values where the sine value is constructible, able to be generated using straightedge and compass, because otherwise we don't really know it without computing Pi first, which would be circular. So we can't use n = 7, 8, 9, and 11.

Comments

Write a comment...