Appendix C: Answers to Even-Numbered Problems in "Games and Information, Second Edition"

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This appendix contains answers to the even-numbered problems in the second edition of *Games and Information* by Eric Rasmusen, published in 1994.

CHAPTER 1

1.2: Nash and Iterated Dominance. (1.2a) Show that every iterated dominance equilibrium s^* is Nash.

Answer. Suppose that s^* is not Nash. This means that there exists some i and s_i' such that i could profitably deviate, i.e., $\pi_i(s^*) \leq \pi_i(s_i', s_{-i}^*)$. But that means that once iterated deletion reached the point where the only strategies left for player i were s_i^* and s_i', it could not be true that s_i^* strongly dominated s_i'. Therefore, iterated deletion could not possibly reach s^*.

(1.2b) Show by counterexample that not every Nash equilibrium can be generated by iterated dominance.

Answer. In "Ranked Coordination" (Table 1.7) no strategy can be eliminated by dominance, and the boldfaced strategies are Nash. (question 1.2c. This question should indeed be changed on p. 33 of the book.)

(1.2c) Is every iterated dominance equilibrium made up of weakly dominant strategies?

Answer. No. Look at the "Battle of the Bismarck Sea" (Table 1.2). *North* is not weakly dominant for Kenney. A strategy that is in the equilibrium strategy profile might be a bad reply to some strategies that iterated deletion removed from the original game.

1.4: Pareto Dominance.[1] (1.4a) If a strategy profile s^* is a dominant strategy equilibrium, does that mean it weakly pareto-dominates all other strategy profiles?

Answer. No— think of the "Prisoner's Dilemma", in Table 1.1. (*Confess, Confess*) is a dominant strategy equilibrium, but it does not weakly pareto-dominate (*Deny, Deny*)

(1.4b) If a strategy profile s strongly pareto-dominates all other strategy profiles, does that mean it is a dominant strategy equilibrium?

Answer. No— think of "Ranked Coordination", in Table 1.7. (*Large, Large*) strongly pareto-dominates all other strategy profiles, but is not a dominant strategy equilibrium.[2]

(1.4c) If s weakly Pareto-dominates all other strategy profiles, must it be a Nash equilibrium?

Answer. Yes. If s is weakly pareto-dominant, then $\pi_i(s) \geq \pi_i(s'), \forall s', \forall i$. If s is Nash, $\pi_i(s) \geq \pi_i(s_i', s_{-i}), \forall s_i', \forall i$. Since $\{s_i', s_{-i}\}$ is a subset of $\{s'\}$, if s satisfies the condition to be weakly pareto-dominant, it must also be a Nash equilibrium.

1.6: Drawing Outcome Matrices. It can be surprisingly difficult to look at a game using new notation. In this exercise, redraw the outcome matrix in a different form than in the main text. In each case, read the description of the game and draw the outcome matrix as instructed. You will learn more if you do this from the description, without looking at the conventional outcome matrix.

(1.6a) "The Battle of the Sexes" (Table 1.6). Put (*Prize Fight, Prize Fight*) in the northwest corner, but make the woman the row player.

Answer. See Table C.1.

Table C.1 "Rearranged Battle of the Sexes I"

		Man		
		Prize Fight		*Ballet*
	Prize Fight	**1,2**	←	-5,-5
Woman:		↑		↓
	Ballet	-1,-1	→	**2,1**

Payoffs to: (Woman, Man).

(1.6b) "The Prisoner's Dilemma" (Table 1.1). Put (*Confess, Confess*) in the northwest corner.

Answer. See Table C.2.

Table C.2 "Rearranged Prisoner's Dilemma"

		Column		
		Confess		*Deny*
	Confess	**-8,-8**	←	0,-10
Row:		↑		↑
	Deny	-10,0	←	-1,-1

Payoffs to: (Row, Column).

(1.6c) "The Battle of the Sexes" (Table 1.6). Make the man the row player, but put (*Ballet, Prize Fight*) in the northwest corner.

Answer. See Table C.3.

Table C.3 "Rearranged Battle of the Sexes II"

		Woman		
		Prize Fight		*Ballet*
	Ballet	-5,-5	→	**1,2**
Man:		↓		↑
	Prize Fight	**2,1**	←	-1,-1

Payoffs to: (Man, Woman).

CHAPTER 2

2.2: Elmer's Appetite. Mrs. Jones has made an apple pie for her son, Elmer, and she is trying to figure out whether the pie tasted divine, or merely good. Her pies turn out divinely a third of the time. Elmer might be ravenous, or merely hungry, and he will eat either 2, 3, or 4 pieces of pie. Mrs. Jones knows he is ravenous half the time (but not which half). If the pie is divine, then if Elmer is hungry the probabilities of the three consumptions are (0, 0.6, 0.4), but if he is ravenous the probabilities are (0,0,1). If the pie is just good, the probabilities are (0.2, 0.4, 0.4) if he is hungry and (0.1,0.3,0.6) if he is ravenous.

Elmer is a sensitive, but useless, boy. He will always say that the pie is divine and his appetite weak, regardless of his true inner feelings.

(2.2a) What is the probability that he will eat 4 pieces of pie?

Answer. P(4) = 17/30 (about 0.57) $(= P(4|Divine)P(Divine) + P(4|Good)P(Good) = ((\frac{1}{2}) \cdot 0.4 + (\frac{1}{2}) \cdot 1)(\frac{1}{3}) + ((\frac{1}{2}) \cdot 0.4 + (\frac{1}{2}) \cdot 0.6))(\frac{2}{3})$.

(2.2b) If Mrs. Jones sees Elmer eat 4 pieces of pie, what is the probability that he is ravenous and the pie is merely good?

Answer. $P(Ravenous, Good|4) = 6/17$ (about 0.35) $(= \frac{P(4|RG)P(RG)}{P(4)} = \frac{0.6((\frac{1}{2})\cdot(\frac{2}{3}))}{(17/30)})$.

(2.2c) If Mrs. Jones sees Elmer eat 4 pieces of pie, what is the probability that the pie is divine?

Answer. $P(Divine|4) = 7/17$ (about 0.41) $(= \frac{P(4|D)P(D)}{P(4)} = \frac{((\frac{1}{2})\cdot 0.4+(\frac{1}{2})\cdot 1)(\frac{1}{3})}{17/30})$.

2.4: The Battleship Problem. The Pentagon has the choice of building one battleship or two cruisers. One battleship costs the same as two cruisers, but a cruiser is sufficient to carry out the navy's mission— if the cruiser survives to get close enough to the target. The battleship has a probability of p of carrying out its mission, whereas a cruiser only has probability $p/2$. Whatever the outcome, the war ends and any surviving ships are scrapped. Which option is superior?[3]

Answer. The battleship completes its mission with probability p. Each cruiser is sunk with probability $1 - p/2$, so both are sunk with probability $(1-p/2)^2$, and then the mission fails. Hence, at least one cruiser survives to complete the mission with probability $1-(1-p/2)^2$, which equals $p - (p^2/4)$, which is less than p. Therefore, buy the battleship.

2.6: California Drought. California is in a drought and the reservoirs are running low. The probability of rainfall in 1991 is $\frac{1}{2}$, but with probability 1 there will be heavy rainfall in 1992. The state uses rationing rather than the price system, and it must decide how much water to consume in 1990, and how much to save till 1991. Each Californian has a utility function of

Answer. Suppose the total amount of existing water is $\overline{w}$ units. Expected utility from this water is $U = 0.5log(w_{90}) + 0.5[log(w_{90}) + log(\overline{w} - w_{90})]$, since if it rains in 1991 the saved water won't be needed. Differentiating with respect to w_{90} and equating to zero gives $w^*_{90} = (\frac{2}{3})\overline{w}$. Thus, two thirds of the water should be consumed the first year.

CHAPTER 3

3.2: Running from the Gestapo. Two risk-neutral men, Schmidt and Braun, are walking south along a street in Nazi Germany when they see a single Gestapo agent coming to check their papers. Only Braun has his papers (unknown to the Gestapo, of course). The Gestapo agent will catch both men if both or neither of them run north, but if just one runs, he must choose which one to stop— the walker or the runner. The penalty for being without papers is 24 months in prison. The penalty for running away from an agent of the state is 6 months in prison. The two friends want to maximize their joint welfare, which the Gestapo man wants to minimize. Braun moves first, then Schmidt, then the Gestapo.

(3.2a) What is the outcome matrix for outcomes that might be observed in equilibrium? (Use θ for the probability the Gestapo chases the runner and γ for the probability Braun runs.)

Answer. We can rule out both Schmidt and Braun either walking or running, by iterated dominance. The first strategy the Gestapo would eliminate is: "Stand still while both men go away together." Once that is eliminated, Schmidt and Braun find it weakly dominant to split their actions. If Braun runs, Schmidt will walk; if Braun walks, Schmidt will run. Otherwise, the Gestapo man will catch both. The expected penalty from running and being caught, aside from any penalty for not having papers, is 6 months in prison (=(1/4) 24). The payoffs to the friends from the various remaining outcomes are shown in Table C.4.

Table C.4 Running from the Gestapo

	Gestapo chases runner (θ)	Gestapo chases walker ($1-\theta$)
Braun runs (γ)	-6	-24
Schmidt runs ($1-\gamma$)	-30	0

(3.2b) What is the probability that the Gestapo agent chases the runner (call it θ^*)?

Answer. There is no pure strategy equilibrium. In the mixed-strategy equilibrium, the friends' payoffs must be equal from the two pure strategies. $\pi(Braun\ runs) = \pi(Schmidt\ runs)$, so

(3.2c) What is the probability that Braun runs (call it γ^*)?

<u>*Answer.*</u> In the mixed-strategy equilibrium, the Gestapo payoffs must be equal from the two pure strategies: $\pi(chase\ runner) = \pi(chase\ walker)$, so

$$\gamma(6) + (1-\gamma)(30) = \gamma(24) + (1-\gamma)(0). \quad (2)$$

Therefore, $\gamma^* = \frac{5}{8}$.

(3.2d) Since Schmidt and Braun share the same objectives, is this a cooperative game?

<u>*Answer.*</u> No. Their utility function is the same, and the model is set up so that they can take the same action without coordination difficulties, but the modeller is not trying to discover what sort of binding agreement they might make with the Gestapo.[5]

A Goethe University PhD-student group pointed out to me that this is actually a game of incomplete information, a topic not yet covered by Chapter 3 of the book. The Gestapo man does not know the payoffs of the game, because he does not know whether his payoff is higher from chasing Schmidt or Braun. They know, so it is also a game of incomplete information.

We do not need the special techniques of incomplete information, though, because this game can also be modelled as a simultaneous game of complete information. Think of the actions as (Gestapo: "Chase the Runner" or "Chase the Walker") (S-B: "Man with Papers Runs", "Man without Papers Runs").

3.4: Mixed Strategies in "The Battle of the Sexes". Refer back to "The Battle of the Sexes" and "Ranked Coordination" in Section 1.4. Denote the probabilities that the man and woman pick *Prize Fight* by γ and θ.

(3.4a) Find an expression for the man's expected payoff.

<u>*Answer.*</u> $\pi_m = \gamma(2\theta + (0)[1-\theta]) + (1-\gamma)((0)\theta + 1[1-\theta])$.

(3.4b) Find the first order condition for the man's choice of strategy.

<u>*Answer.*</u> $\frac{d\pi_m}{d\gamma} = 2\theta - [1-\theta]) = 0$.

(3.4c) What are the equilibrium values of γ and θ, and the expected payoffs?

<u>*Answer.*</u> $\theta^* = 1/3, \gamma^* = 2/3$, $\pi_m = \frac{2}{3}, \pi_w = \frac{2}{3}$.

(3.4d) Find the most likely outcome and its probability.

<u>*Answer.*</u> *(Prize Fight, Ballet)* for (M, W), which has probability 4/9, about .444.

(3.4e) What is the equilibrium payoff in the mixed strategy equilibrium for "Ranked Coordination"?

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[5] This models a joke that continues like this: Braun was more than a little overweight

Answer. The probability is found by solving $2\theta + (1-\theta)(-1) = (-1)\theta + (1-\theta)$. Therefore, $Prob(Large) = 2/5$, $\pi = 1/5$ for each player.

(3.4f) Why is the mixed strategy equilibrium a better focal point in "The Battle of the Sexes" than in "Ranked Coordination"?

Answer. "Ranked Coordination" has a Pareto-dominant Nash equilibrium, so if players are optimistic, they will focus on that equilibrium. In "The Battle of the Sexes", neither Nash equilibrium is pareto-dominant.

3.6: Alba and Rome: Asymmetric information and mixed strategies. A Roman, Horatius, unwounded, is fighting the three Curiatius brothers from Alba, each of whom is wounded. If Horatius continues fighting, he wins with probability 0.1, and the payoffs are (10,-10) for (Horatius, Curiatii) if he wins, and (-10,10) if he loses. With probability $\alpha = 0.5$, Horatius is panic-stricken and runs away. If he runs and the Curiatii do not chase him, the payoffs are (-20, 10). If he runs and the Curiatius brothers chase and kill him, the payoffs are (-21, 20). If, however, he is not panic-stricken, but he runs anyway and the Curiatii give chase, he is able to kill the fastest brother first and then dispose of the other two, for payoffs of (10,-10). Horatius is, in fact, not panic-stricken.

(3.6a) With what probability θ would the Curiatii give chase if Horatius were to run?

Answer. In a mixed-strategy equilibrium,

$$\pi_h(run) = \pi_h(not\ run), \tag{3}$$

so

$$\theta(10) + (1-\theta)(-20) = 0.1(10) + 0.9(-10) \tag{4}$$

and $\theta^* = \frac{12}{30} = 0.4$.

(3.6b) With what probability γ does Horatius run?

Answer. In a mixed-strategy equilibrium,

$$\pi_c(chase) = \pi_c(not\ chase), \tag{5}$$

so

$$\begin{aligned}\alpha(20) + (1-\alpha)\gamma(-10) + (1-\alpha)(1-\gamma)[0.1(-10) + 0.9(10)] = \\ \alpha(10) + (1-\alpha)\gamma(10) + (1-\alpha)(1-\gamma)[0.1(-10) + 0.9(10)],\end{aligned} \tag{6}$$

which reduces to

$$20\alpha - 10\gamma + 10\alpha\gamma = 10\alpha + 10\gamma - 10\alpha\gamma, \tag{7}$$

so $\gamma^* = \frac{\alpha}{2-2\alpha}$, which equals 0.5 if $\alpha = 0.5$.

(3.6c) How would θ and γ be affected if the Curiatii falsely believed that the probability of Horatius being panic-stricken was 1? What if they believed it was 0.9?

4.2: Evolutionarily Stable Strategies A population of scholars are playing the following coordination game over their two possible conversation topics over lunch, football and economics. Let $N_t(F)$ and $N_t(E)$ be the numbers who talk football and economics in period t, and θ be the percentage who talk football, so $\theta = \frac{N(football)}{N(football)+N(economics)}$. Government regulations requiring lunchtime attendance and stipulating the topics of conversation have maintained the values $\theta = 0.5$, $N_t(F) = 50,000$ and $N_t(E) = 50,000$ up to this year's deregulatory reform. In the future, some people may decide to go home for lunch instead, or change the conversation. Table 4.5 shows the payoffs.

Table 4.5 Evolutionarily Stable Strategies

		Scholar 2	
		Football (θ)	*Economics* $(1-\theta)$
	Football (θ)	1,1	0, 0
Scholar 1			
	Economics $(1-\theta)$	0, 0	5,5

Payoffs to: (Scholar 1, Scholar 2).

(4.2a) There are three Nash equilibria: *(Football, Football)*, *(Economics, Economics)*, and a mixed-strategy equilibrium. What are the evolutionarily stable strategies?

Answer. *Football* and *Economics*. The mixed strategy can be invaded by either of these, since each strategy does very well against itself, and also does well in the mixed population.

(4.2b) Let $N_t(s)$ be the number of scholars playing a particular strategy in period t and let $\pi_t(s)$ be the payoff. Devise a Markov difference equation to express the population dynamics from one period to the next: $N_{t+1}(s) = f(N_t(s), \pi_t(s))$. Start the system with a population of 100,000, half the scholars talking football and half talking economics. Use your dynamics to finish Table 4.6.

Answer. One of the many possible dynamics is $N_{t+1}(s) = N_t(s)[0.4\pi_t(s)]$. This is shown in Table C.5.

Table C.5 Markov Conversation Dynamics

t	$N_t(F)$	$N_t(E)$	θ	$\pi_t(F)$	$\pi_t(E)$
-1	50,000	50,000	0.5	0.5	2.5
0	50,000	50,000	0.5	0.5	2.5
1	10,000	50,000	0.167	0.167	4.167

(4.2c) Repeat part (b), but specifying non-Markov dynamics, in which $N_{t+1}(s) = f(N_t(s), \pi_t(s), \pi_{t-1}(s))$.

Answer. $N_{t+1}(s) = N_t(s)[0.2\pi_t(s) + 0.2\pi_{t-1}(s)]$. This is shown in Table C.6.

Table C.6 Non-Markov Conversation Dynamics

t	$N_t(F)$	$N_t(E)$	θ	$\pi_t(F)$	$\pi_t(E)$
-1	50,000	50,000	0.5	0.5	2.5
0	50,000	50,000	0.5	0.5	2.5
1	10,000	50,000	0.167	0.167	2.5
2	1,333	66,667	0.020	0.02	4.90

4.4: "Grab the Dollar". Table 4.7 shows the payoffs for the simultaneous-move game, "Grab the Dollar". A silver dollar is put on the table between Smith and Jones. If one grabs it, he keeps the dollar, for a payoff of 4 utils. If both grab, then neither gets the dollar, and both feel bitter. If neither grabs, each gets to keep something.

Table 4.7 "Grab the Dollar"

		Jones	
		Grab (θ)	*Wait* $(1-\theta)$
Smith:	*Grab* (θ)	$-1, -1$	$4, 0$
	Wait $(1-\theta)$	$0, 4$	1,1

Payoffs to: (Smith, Jones).

(4.4a) What are the evolutionarily stable strategies?

Answer. The ESS is mixed and unique. Let $Prob(Grab) = \theta$. Then $\pi(Grab) = -1(\theta) + 4(1-\theta) = \pi(Wait) = 0(\theta) + 1(1-\theta)$, which solves to $\theta = 3/4$. Three fourths of the population plays *Grab*.

(4.4b) Suppose each player in the population is a point on a continuum, and the initial amount of players is 1, evenly divided between *Grab* and *Wait*. Let $N_t(s)$ be the amount of players playing a particular strategy in period t and let $\pi_t(s)$ be the payoff. Let the population dynamics be $N_{t+1}(i) = (2N_t(i))\left(\frac{\pi_t(i)}{\sum_j \pi_t(j)}\right)$. Find the missing entries in Table 4.8.

Answer. See Table C.7.

Table C.7 "Grab the Dollar": Dynamics I

t	$N_t(G)$	$N_t(W)$	$N_t(total)$	θ	$\pi_t(G)$	$\pi_t(w)$
0	0.5	0.5	1	0.5	1.5	0.5
1	0.75	0.25	1	0.75	0.25	0.25

Table C.8 "Grab the Dollar": Dynamics II

t	$N_t(G)$	$N_t(W)$	$N_t(total)$	θ	$\pi_t(G)$	$\pi_t(w)$
0	.5	0.5	1	.5	1.5	0.5
1	1.75	1.25	3	0.58	1.1	0.42
2	6.03	3.19	9.22	0.65	0.75	0.35

(4.4d) Which three games that have appeared so far in the book resemble "Grab the Dollar"?

Answer. "Chicken", "The Battle of the Sexes", and "The Hawk-Dove Game". (Question 4.4d should be added to the book on page 119.)

CHAPTER 5

5.2: Product Quality with Lawsuits. Modify the game "Product Quality" (Section 5.8) by assuming that if the seller misrepresents his quality he must, as a result of a class-action suit, pay damages of x per unit sold, where $x \in (0, c]$ and the seller becomes liable for x at the time of sale.

(5.2a) What is $\tilde{p}$ as a function of $x, F, c,$ and r? Is $\tilde{p}$ greater than when $x = 0$?

Answer. To avoid low quality, it must be that $\frac{q(p-x)}{1+r} = \frac{q(p-c)}{r}$,and so $rqp - rqx = qp - qc + rqp - rqc$, and $0 = rx + p - c - rc$, and $\tilde{p} = (1+r)c - rx$. Thus, the threat of a lawsuit reduces the quality-ensuring price.

(5.2b) What is the equilibrium output per firm? Is it greater than when $x = 0$?

Answer. $\frac{q_i((1+r)c - rx - c)}{r} = F$, for zero profits ex ante. Thus, $q_i(c + rc - rx - c) = rF$, and $q_i = \frac{F}{c-x}$. The equilibrium output has increased because $x > 0$.

(5.2c) What is the equilibrium number of firms? Is it greater than when $x = 0$?

Answer. For supply to equal demand, $\frac{nF}{c-x} = q(p)$, so that

$$\tilde{n} = \frac{(c-x)q(\tilde{p})}{F}. \qquad (8)$$

This might either increase or decrease in x, because $(c - x)$ decreases in x, but $\tilde{p}$ decreases also, so $q(\tilde{p})$ increases when $x > 0$.

(5.2d) If, instead of x per unit, the seller pays X to a law firm to successfully defend him, what is the incentive compatibility constraint?

Answer. The incentive compatibility constraint, $\pi_{low} \leq \pi_{high}$, becomes

$$q_i p \quad X \quad \leq q_i(p-c)$$

which must be solved simultaneously with the other equations to find p and q_i.

5.4: Repeated Entry Deterrence. Assume that "Entry Deterrence I" is repeated an infinite number of times, with a tiny discount rate and with payoffs received at the start of each period. In each period, the entrant chooses *Enter* or *Stay out*, even if he entered previously.

(5.4a) What is a perfect equilibrium in which the entrant enters each period?

Answer. $(Enter, Collude)$ each period.

(5.4b) Why is (*Stay out, Fight*) not a perfect equilibrium?

Answer. $(Stay\ out, Fight|Enter)$ gives the incumbent no incentive to choose $Fight$. Given the entrant's strategy, if somehow the game ends up off the equilibrium path with the entrant having entered, the entrant will *Stay Out* in succeeding periods. Hence, the incumbent would deviate by choosing $Collude$ and getting 50 instead of 0.

(5.4c) What is a perfect equilibrium in which the entrant never enters?

Answer. *Entrant: Stay out* unless the incumbent has chosen $Collude$ in some previous period, in which case, $Enter$.

Incumbent: $Fight|Enter$ unless the incumbent has chosen $Collude$ in some previous period, in which case, choose $Collude|Enter$.

In this equilibrium, the incumbent suffers a heavy penalty if he ever colludes.

(5.4d) What is the maximum discount rate for which your strategy profile in part (c) is still an equilibrium?

Answer. If the discount rate is too high, the entrant will enter and the incumbent will prefer to collude. Suppose the entrant has entered, and the incumbent has never yet colluded. The incumbent's choice is between

$$\pi(collude) = 50 + \frac{50}{r} \tag{11}$$

and

$$\pi(fight) = 0 + \frac{100}{r} \tag{12}$$

These two payoffs equal each other if $r = 1$, so if the discount rate is anything less, the equilibrium in (c) remains an equilibrium.

CHAPTER 6

6.2: Limit Pricing.[6] An incumbent firm operates in the local computer market, which is a natural monopoly in which only one firm can survive. The monopoly can price *Low*, losing 40 in profits, or *High*, losing nothing.

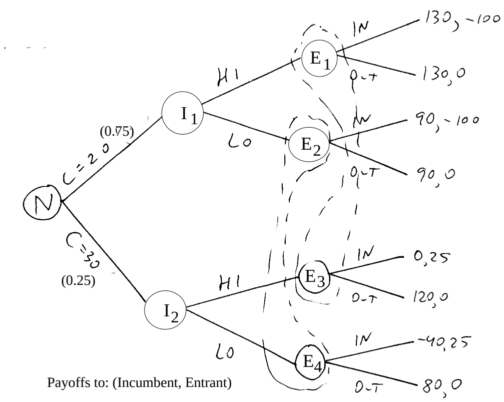
its operating cost of 25 is common knowledge. The firm with the highest operating cost immediately drops out if it has a competitor, and the survivor earns the monopoly revenue of 150.

(6.2a) Draw the extensive form for this game.

Answer. See Figure C.1.

Figure C.1 The Extensive Form for "Limit Pricing"

Figure A1.7: The Extensive Form for ''Limit Pricing''



(6.2b) Why is there no perfect equilibrium in which the incumbent prices *Low* only when its costs are 20? (no "separating" equilibrium)

Answer. In any such separating equilibrium, the entrant would not enter if the incumbent priced *Low*. Therefore, the incumbent would price *Low* in the first period even if $C = 30$, for a payoff of 80 instead of the 0 payoff from pricing *High* and inducing entry. This strategy of $Low|30$ contradicts the postulated separating equilibrium. Note also that if it cost 125 to price *Low*, a separating equilibrium would indeed exist, because the incumbent would be unwilling to pay 125 to fool the entrant into thinking that $C = 20$ when really $C = 30$.

(6.2c) In a perfect bayesian equilibrium in which the incumbent prices *Low* regardless of its costs (a "pooling" equilibrium), about what do out-of-equilibrium beliefs have to be specified?

Answer. Beliefs must be specified for the probability that $C = 20$ given that the incumbent prices *High*.

(6.2d) What are two different perfect bayesian equilibria for this game?

Answer. From (b), both equilibria must be pooling.

$$(i)\{(Low|20, Low|30), Out|Low, In|High, Prob(20|High) = 0\}$$
$$(ii)\{(High|20, High|30), Out|Low, Out|High, Prob(20|Low) = 0.5\}.$$

(6.2e) What is a set of out-of-equilibrium beliefs that do not support a pooling equilibrium at a *Low* price?

Answer. $Prob(20|High) = 1$. If this is the belief, then the incumbent with a cost of 20 would deviate to a *High* price, increasing his

(6.4a) Why does $Pr(Strong|Enter, Nature\ said\ nothing) = 0.95$ not support the equilibrium in Section 6.3?

<u>*Answer.*</u> Under these beliefs, if the entrant deviates and enters, the incumbent's expected payoff from $Fight$ is 15 (= 0.95(0)+0.05(300)), which is less than the 50 he can get from $Collude$.

(6.4b) Why is the equilibrium in Section 6.3 not an equilibrium if 0.7 is the probability that Nature tells the incumbent?

<u>*Answer.*</u> The entrant would deviate to $Enter|Strong$. If the entrant is strong, he expects the incumbent to fight with probability 0.3 and collude with probability 0.7. The payoff from entry is then 25 (= $0.3(-10) + 0.7(40)$), which is greater than the 0 from staying out.

(6.4c) Describe the equilibrium if 0.7 is the probability that Nature tells the incumbent. For what out-of-equilibrium beliefs does this remain the equilibrium?

<u>*Answer.*</u> The equilibrium when Nature tells with probability 0.7 is in mixed strategies, because in a pure-strategy equilibrium the incumbent could deduce the entrant's type from whether the entrant enters or not. If only strong entrants entered, the incumbent would never fight entry, and weak entrants would also enter. The equilibrium is

Entrant: $Enter|Strong$
$Enter\ with\ probability\ \theta = 0.2|Weak$

Incumbent: *Collude|(Enter, Nature said "Strong")*
Fight—(Enter, Nature said "Weak"),
$Collude\ with\ probability\ \gamma = 17/22|(Enter, Nature\ said\ nothing)$

The strong entrant enters because his expected payoff is

$$\begin{aligned} \pi_e(Enter|Strong) &= 0.7(40) + 0.3(\gamma(40) + (1-\gamma)(-10)) \\ &= 28 + 12(17/22) - 3(5/22) \\ &> 0. \end{aligned} \tag{13}$$

The weak entrant must be indifferent between entering and staying out, so

$$\pi_e(Enter|Weak) = 0.7(-10)+0.3(\gamma(40)+(1-\gamma)(-10)) = \pi_e(Stay\ out|Weak) = 0, \tag{14}$$

which when solved yields $\gamma = 17/22$.

If the incumbent observes that the entrant has entered, he knows that the entrant might be either strong (probability 0.5) or weak (probability 0.5θ). Using Bayes's Rule and equating the incumbent's payoffs from fighting and colluding gives

Since there is no behavior that could never be observed in in equilibrium, no out-of-equilibrium beliefs need be specified.

CHAPTER 7

7.2: The Principal-Agent Problem. Suppose the agent has the utility function $U = \sqrt{w} - e$, where e can take the levels 0 or 7, and a reservation utility of $\overline{U} = 4$. The principal is risk-neutral. Denote the agent's wage conditioned on output as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 1000. Only the agent observes his effort. Principals compete for agents. Table 7.6 shows the output.

Table 7.6 Output from Low and High Effort

	Probability of Outputs		
Effort	0	1000	Total
Low $(e = 0)$	0.9	0.1	1
High $(e = 7)$	0.2	0.8	1

(7.2a) What are the incentive compatibility, participation, and zero-profit constraints for obtaining high effort?

Answer. The incentive compatibility constraint is

$$0.2\sqrt{\underline{w}} + 0.8\sqrt{\overline{w}} - 7 \geq 0.9\sqrt{\underline{w}} + 0.1\sqrt{\overline{w}}. \qquad (16)$$

The participation constraint is

$$0.2\sqrt{\underline{w}} + 0.8\sqrt{\overline{w}} - 7 \geq 4. \qquad (17)$$

The zero-profit constraint is

$$0.2\underline{w} + 0.8\overline{w} = 0.2(0) + 0.8(1000) = 800. \qquad (18)$$

(7.2b) What would utility be if the wage were fixed and could not depend on output or effort?

Answer. Effort would be low, so the expected value of output would be 100 $(= 0.9(0) + 0.1(1000))$. The wage would be a flat 100, and utility would be 10 $(= \sqrt{100} - 0)$.

(7.2c) What is the optimal contract? What is the agent's utility?

Answer. The participation constraint is not binding, since principals compete for agents. The incentive compatibility constraint can be rewritten, since it is binding, as

$$\sqrt{\overline{w}} - \sqrt{\underline{w}} = 7/0.7 = 10 \qquad (19)$$

Substituting the zero-profit constraint into the incentive compatibility constraint yields

$$\sqrt{\overline{w}} - \sqrt{4000 - 4\overline{w}} = 10. \tag{21}$$

If $\overline{w} = 900$ and $\underline{w} = 400$, then since $30 - \sqrt{4000 - 3600} = 10$, we have a solution. The agent's utility is 21, from

$$0.2\sqrt{400} + 0.8\sqrt{900} - 7 = 21. \tag{22}$$

(7.2d) What would the agent's utility be under full information? Under asymmetric information, what is the agency cost (the lost utility) as a percentage of the utility the agent receives?

Answer. Under full information, the agent would be perfectly insured and would choose high effort. To see that he would choose high effort, note that his wage would be 100 and his utility would be 10 ($= \sqrt{100} - 0$) under low effort. Under high effort, his wage would be 800 ($= 0.2(0) + 0.8(1000)$), and his utility would be 21.3 ($= \sqrt{800} - 7$). Since the utility under asymmetric information is 21, and the difference is 0.3, the agency cost is 0.3/21, about 1.4 percent .

7.4: Bankruptcy Constraints. A risk-neutral principal hires an agent with utility function $U = w - e$ and reservation utility $\overline{U} = 5$. Effort is either 0 or 10. There is a bankruptcy constraint: $w \geq 0$. Output is given by Table 7.8.

Table 7.8 Bankruptcy

	Probability of Outputs		
Effort	0	400	Total
Low ($e = 0$)	0.5	0.5	1
High ($e = 10$)	0.1	0.9	1

(7.4a) What would be the agent's effort choice and utility if he owned the firm?

Answer. $e = 10$, because expected output is then 360 instead of the 200 with low effort, and the agent's utility is 350 instead of 200.

(7.4b) If agents are scarce and principals compete for them what will be the agent's contract under full information? His utility?

Answer. Effort is high, as found in part (a). The wage is 360 for high effort and 0 for low (though there are other possibilities). Agent utility is 350.

(7.4c) If principals are scarce and agents compete to work for them, what will the contract be under full information? What will the agent's utility be?

will the payoffs be for each player?

Answer. An efficiency wage must be paid so that the incentive compatibility constraint of part (d) is satisfied. The participation constraint is thus not binding. The low wage will be 0, since the principal wants to make the gap as big as possible between the low wage and the high wage. The high wage must equal 25 to get incentive compatibility. Hence,

$$U = 0.1(0) + 0.9(25) - 10 = 12.5 \tag{23}$$

$\pi(H) = 337.5 (= 0.1(0-0) + 0.9(400-25))$. This exceeds $\pi(L) = 195 (= 0.5(0-5) + 0.5(400-5))$.

(7.4e) Suppose there is no bankruptcy constraint. If principals are the scarce factor and agents compete to work for them, what will the contract be when the principal cannot observe effort? What will the payoffs be for principal and agent?

Answer. Since agents are risk neutral, selling the store works well. The expected wage must be 15 for the agent so that $U = \overline{U} = 5$, and an incentive compatibility constraint must be satisfied to obtain high effort:

$$0.5w(0) + 0.5w(400) \leq 0.1w(0) + 0.9w(400) - 10, \tag{24}$$

which can be rewritten as $w(400) - w(0) \geq 25$. Many contracts can ensure this. One is to sell the store for 360 minus 10 for the high effort minus 5 for the opportunity cost, which is equivalent to letting the agent keep all the output for a lump-sum payment of 345: $w(0) = 0 + 15 - 360 = -345$ and $w(400) = 400 + 15 - 360 = 55$, which averages to an expected wage of 15 and an expected utility of 5. The principal's payoff is 345.

CHAPTER 8

8.2: Monitoring with Error: Second Offenses.[7] Individuals who are risk-neutral must decide whether to commit zero, one, or two robberies. The cost to society of robbery is 10, and the benefit to the robber is 5. No robber is ever convicted and jailed, but the police beat up any suspected robber they find. They beat up innocent people mistakenly sometimes, as shown by Table 8.2, which shows the probabilities of zero or more beatings for someone who commits zero, one, or two robberies.

Table 8.2 Crime

	Beatings		
Robberies	0	1	2
0	0.81	0.18	0.01
1	0.60	0.34	0.06
2	0.49	0.42	0.09

Answer. If $\pi(n)$ is the payoff from committing n robberies, the payoffs are

$$\begin{aligned} \pi(0) &= -0.18p^* - 0.01(2p^*) = -0.2p^* \\ \pi(1) &= 5 - 0.34p^* - 0.06(2p^*) = 5 - 0.46p^* \\ \pi(2) &= 10 - 0.42p^* - 0.09(2p^*) = 10 - 0.6p^* \end{aligned} \quad (25)$$

Incentive compatibility requires that $\pi(0) \geq \pi(1)$ and $\pi(0) \geq \pi(2)$, so $-0.2p^* \geq 10 - 0.6p^*$ and $-0.2p^* \geq 5 - 0.46p^*$. These are solved by $p^* \geq 19.2$ (rounded) and $p^* \geq 25$, so since both inequalities must be true, $p^* = 25$.

Note that the problem narrows the issue to policies which completely deter robbery. The optimal level of robbery is a different issue.

(8.2b) In equilibrium, what percentage of beatings are of innocent people? What is the payoff of an innocent man?

Answer. 100 percent of the beatings are of innocent people. The payoff of an innocent person is -5 ($= \pi(0) = -0.2(25)$).

(8.2c) Now consider a more flexible policy, which inflicts heavier beatings on repeat offenders. If such flexibility is possible, what are the optimal severities for first and second time offenders? (call these p_1 and p_2) What is the expected utility of an innocent person under this policy?

Answer. One of the incentive compatibility constraints is that $\pi(0) \geq \pi(1)$, so $-0.18p_1 - 0.01(p_1 + p_2) \geq 5 - 0.34p_1 - 0.06(p_1 + p_2)$. The problem is to maximize $-0.18p_1 - 0.01(p_1 + p_2)$ subject to that constraint, which can be rewritten as $0.21p_1 + 0.05p_2 \geq 5$. Since the maximand is linear in the control variables, the solution will be a corner solution with a binding constraint. Increasing p_1 by one unit subtracts 0.19 from the maximand while adding 0.21 to the constraint, a ratio of about -0.9, while increasing p_2 yields a ratio of -0.2, so $p_1 = 0$ in the solution. Solving the problem subject to the incentive compatibility constraint that $\pi(0) \geq \pi(2)$ also would give $p_1 = 0$ because there, too, the best way for the government to put a wedge between the payoff of the innocent and the guilty is by focussing on second offenses.

Since $p_1 = 0$, the payoffs are

$$\begin{aligned} \pi(0) &= -0.01p_2, \\ \pi(1) &= 5 - 0.06p_2, \\ \pi(2) &= 10 - 0.09p_2. \end{aligned} \quad (26)$$

Again, committing two robberies is the most tempting deviation, so the binding incentive compatibility constraint is $-0.01p_1 = 10 - 0.09p_2$, which yields $p_1 = 125$, a severe beating indeed. The payoff of an innocent man is -1.25 ($= \pi(0) = -0.01(125)$), higher than with penalty p^*.

(8.2d) Suppose that the probabilities are as given in Table 8.3. What

	Beatings		
Robberies	0	1	2
0	0.9	0.1	0
1	0.6	0.3	0.1
2	0.5	0.3	0.2

Answer. Since there is no chance of an innocent person being falsely accused twice, the optimal penalty is 0 for the first offense and 1,000 (or some other large number) for the second offense. In this case, the limitation of the penalty for first-time offenders to 0 is not a constraint; it arises at the unconstrained optimum.

8.4: Teams. A team of two workers produces and sells widgets for the principal. Each worker chooses high or low effort. An agent's utility is $U = w - 20$ if his effort is high, and $U = w$ if it is low, with a reservation utility of $\overline{U} = 0$. Nature chooses business conditions to be excellent, good, or bad, with probabilities θ_1, θ_2, and θ_3. The principal observes output, but not business conditions, as shown in Table 8.4.

Table 8.4 Team Output

	Excellent (θ_1)	*Good* (θ_2)	*Bad* (θ_3)
$High, High$	100	100	60
$High, Low$	100	50	20
Low, Low	50	20	0

(8.4a) Suppose $\theta_1 = \theta_2 = \theta_3$. Why is $\{(w(100) = 30, w(not\ 100) = 0), (High, High)\}$ not an equilibrium?

Answer. A worker would deviate. His payoff from $High$ is $\pi(High) = \frac{2}{3}(30) - 20 = 0$, and his payoff from Low is $\pi(Low) = \frac{1}{3}(30) = 10 > 0$.

(8.4b) Suppose $\theta_1 = \theta_2 = \theta_3$. Is it optimal to induce high effort? What is an optimal contract with nonnegative wages?

Answer. High effort by both workers is efficient here. The expected output minus the real cost of labor for HH is $46\,\frac{2}{3}$ $(= \frac{2}{3}(100) + \frac{1}{3}(60) - 40)$; from HL it is $36\,\frac{2}{3}$ $(= \frac{1}{3}(100) + \frac{1}{3}(50) + \frac{1}{3}(20) - 20)$; from LL it is $23\,\frac{1}{3}$ $(= \frac{1}{3}(50) + \frac{1}{3}(20))$. An optimal contract is the boiling-in-oil contract $((w = 60|(q = 60), w = 0|(q \neq 60))$. This satisfies the participation constraint by giving each worker an expected utility of 0 $(= \frac{1}{3}(60) - 20)$, and the incentive compatibility constraint by making a worker's expected utility 0 if he chooses low effort.

(8.4c) Suppose $\theta_1 = 0.5$, $\theta_2 = 0.5$, and $\theta_3 = 0$. Is it optimal to induce high effort? What is an optimal contract (possibly with negative wages)?

Answer. High effort is still efficient. The expected output minus the real cost of labor for (HH) is 60 (=100 - 40); from HL it is 55 $(= \frac{1}{2}(100 + 50) - 20$; from LL it is 35 $(= \frac{1}{2}(50 + 20))$. An optimal contract is $(w = -3000|(q = 50), w = 20|(q \neq 50))$. This satisfies the participation constraint by giving each worker an expected utility of 0 $(= 20 - 20)$, and the incentive compatibility constraint by making a worker's expected utility very negative if he chooses low effort.

(8.4d) Should the principal stop the agents from talking to each other?

Answer. It doesn't matter. If there were multiple equilibria, talk might help the agents coordinate on a preferred equilibrium, but that is not the case here.

CHAPTER 9

9.2: Testing and Commitment. Fraction β of workers are talented, with output $a_t = 5$, and fraction $(1-\beta)$ are untalented, with output $a_u = 0$. Both types have a reservation wage of 1 and are risk-neutral. At a cost of 2 to itself and 1 to the job applicant, employer Apex can test a job applicant and discover his true ability with probability θ, which takes a value of something over 0.5. There is just one period of work. Let $\beta = 0.001$. Suppose that Apex can commit itself to a wage schedule before the workers take the test, and that Apex must test all applicants and pay all the workers it hires the same wage, to avoid grumbling among workers and corruption in the personnel division.

(9.2a) What is the lowest wage, w_t, that will induce talented workers to apply? What is the lowest wage, w_u, that will induce untalented workers to apply? Which is greater?

Answer. If the firm cannot pay different wages depending on whether an applicant passes the test, it will not hire those who fail, since there would be no way to deter the untalented from applying. If the talented worker applies, he pays 1 for the test, his chance of getting the job and the wage w_t is θ, and his chance of not getting the job and settling for the reservation wage of 1 is $(1-\theta)$, so his expected utility equals his reservation utility if $-1 + \theta w_t + (1-\theta)(1) = 1$. Therefore, $w_t = 1 + \frac{1}{\theta}$. The expected utility of the untalented worker who applies equals his reservation utility if $-1 + \theta(1) + (1-\theta)w_u = 1$, so $w_u = \frac{2-\theta}{1-\theta} = 1 + 1/(1-\theta)$. Since $\theta > 0.5$, $w_t < w_u$. Untalented workers need a higher wage if they are to apply, because they fail the test more often.

The firm must pay at least $w = 1 + \frac{1}{\theta}$, the minimum wage acceptable to the talented, and if it pays that wage to those who pass the test, the untalented will not bother to apply.

(9.2b) What is the minimum accuracy value $\underline{\theta}$ that will induce Apex to use the test? What are the firm's expected profits per worker who applies?

Answer. Without testing, the average quality of workers is $0.001(5) + 0.999(0)$, which is less than 1, so the firm will not hire at all if it does not test, and it will test if it can thereby make positive profits.
Apex will offer the wage of $w = 1 + \frac{1}{\theta}$, so only the talented workers will apply, and it will turn down any worker who fails the test. Apex's expected profit per worker who applies is then

$$\pi = -2 + \theta(5 - 1 - \tfrac{1}{\theta}.) \tag{27}$$

This exceeds zero if $\theta > 3/4$. Thus, $\underline{\theta} = 3/4$.

(9.2c) Now suppose that Apex can pay w_p to workers who pass the test at some wage or other. The self selection constraint that induces the talented to apply is

$$-1 + \theta w_p + (1-\theta) w_f \geq 1. \tag{28}$$

The self selection constraint that induces the untalented to not apply is

$$-1 + (1-\theta) w_p + \theta w_f \leq 1. \tag{29}$$

For the talented worker who has taken the test to be willing to take the job it must also be true that $w_p \geq 1$ and $w_f \geq 1$, unless the worker could somehow commit to taking the job once he had taken the test.

Apex will want to minimize the amount paid to the workers it hires, which means that inequality (28) will hold as an equality. There are many equally good wage combinations that do this while satisfying both constraints, all yielding the same profits. For example, let $w_f = 1$. Then from (28), $w_p = 1 + \frac{1}{\theta}$. Profit per worker who applies is

$$\pi \quad = -2 + 5 - \theta(1 - \tfrac{1}{\theta}) - (1-\theta)(1) = 3 \tag{30}$$

This only requires that $\theta > 0.5$, so the test need only be the slightest bit informative to be effective.

(9.2d) What happens if Apex cannot commit to paying the advertised wage, and can decide each applicant's wage individually?

Answer. If Apex cannot commit to wages ahead of time, then it will only pay 1 to a worker who has passed the test. Having already incurred the cost of testing, that cost is sunk for him, and he will accept the wage of 1, which yields a maximum utility of $-1+1=0$, and yields even less if he flunks. Foreseeing this trap, the worker would not apply.

(9.2e) If Apex cannot commit to testing every applicant, why is there no equilibrium in which either untalented workers do not apply or the firm tests every applicant?

Answer. If no untalented workers apply, Apex would deviate and save 2 by skipping the test and hiring every applicant, but then the untalented workers would start applying. If Apex tests every applicant, however, and pays only w_H, then no untalented worker will apply, and, again, Apex would deviate and skip the test.

9.4: Two-Time Losers.[8] Some people are strictly principled and will commit no robberies, even if there is no penalty. Others are incorrigible criminals and will commit two robberies, regardless of the penalty. Society wishes to inflict a certain penalty on criminals as retribution. Retribution requires an expected penalty of 15 per crime (15 if detection is sure, 150 if it has probability 0.1, etc.). Innocent people are sometimes falsely convicted, as shown in Table 9.2.

	Convictions		
Robberies	0	1	2
0	0.81	0.18	0.01
2	0.49	0.42	0.09

Two systems are proposed: (i) a penalty of X for each conviction, and (ii) a penalty of 0 for the first conviction, and some amount P for the second conviction.

(9.4a) What must X and P be to achieve the desired amount of retribution?

Answer. The expected punishment of a criminal under system (i) is $0.42X+0.09(2X) = 0.6X$. This must equal 30, so $X = 50$. The expected punishment of a criminal under system (ii) is $0.42(0)+0.09(P) = 0.09P$. This must equal 30, so $P = 333\,\frac{1}{3}$.

(9.4b) Which system inflicts the lesser cost on innocent people? How much is the cost in each case?

Answer. The expected cost under system (i) is 10 (=$0.18X + 0.01(2X) = 0.2X$). The expected cost under system (ii) is 3.33 (=$0.18(0) + 0.01P$). System (ii) has lower costs.

CHAPTER 10

10.2: Productive Education and Nonexistence of Equilibrium. Change "Education I" so that the two equally likely abilities are $a_L = 2$ and $a_H = 5$ and education is productive: the payoff of the employer whose contract is accepted is $\pi_{employer} = a + 2y - w$. The worker's utility function remains $U = w - \frac{8y}{a}$.

(10.2a) Under full information, what are the wages for educated and uneducated workers of each type, and who acquires education?

Answer. The wages paid will be equal to the worker's productivity: $w_{H0} = 5, w_{H1} = 7, w_{L0} = 2$, and $w_{L1} = 4$.

This being the case, only the high-ability workers will acquire education, because

$$U_H(y = 1) = 7 - 8/5 = 5.4 > U_H(y = 0) = 5 \qquad (31)$$

but

$$U_L(y = 1) = 4 - 8/2 = 0 < U_L(y = 0) = 2. \qquad (32)$$

10.2b) Show that with incomplete information the equilibrium is unique (except for beliefs and wages out of equilibrium) but unreasonable.

Answer. The equilibrium is pooling, with zero education:

employers believe that someone with education has low ability and a productivity of $2+2y$. This pooling equilibrium seems unreasonable because it requires that the employers update their beliefs about someone who acquires education in an odd way: their education reduces the employer's beliefs that they have high ability. Passive conjectures will not support the pooling equilibrium. Under passive conjectures, $w_1 = 5.5(= 0.5(4) + 0.5(7))$, and when the wage is that high, the high-ability workers deviate to acquire education.

There cannot be a pooling equilibrium with $y = 1$, because then the average productivity would be 5.5 and the payoff to the Lows would be 1.5 $(= 5.5 - \frac{8}{2})$, less than the 2 they could get by deviating to $y = 0$. There cannot be a separating equilibrium, because the payoff to the Lows would be 2, which is less than the 3 $(= 7 - \frac{8}{2})$ they could get by deviating to $y = 1$. Thus, the peculiar pooling equilibrium is unique.

10.4: Signalling with a Continuous Signal. Suppose that with equal probability a worker's ability is $a_L = 1$ or $a_H = 5$, and that the worker chooses any amount of education $y \in [0, \infty)$. Let $U_{worker} = w - \frac{8y}{a}$ and $\pi_{employer} = a - w$.

(10.4a) There is a continuum of pooling equilibria, with different levels of y^*, the amount of education necessary to obtain the high wage. What education levels, y^*, and wages, $w(y)$, are paid in the pooling equilibria, and what is a set of out-of-equilibrium beliefs that supports them? What are the incentive compatibility constraints?

Answer. A pooling equilibrium for any $y^* \in [0, 0.25]$ is

$$w = \left| \begin{array}{l} 1 \; if \; y \neq y^* \\ 3 \; if \; y = y^* \end{array} \right. \tag{34}$$

with the out-of-equilibrium belief that $Pr(L|(y \neq y^*)) = 1$, and with $y = y^*$ for both types.

The self selection constraints say that neither High nor Low workers want to deviate by acquiring other than y^* education. The most tempting deviation is to zero education, so the constraints are:

$$U_L(y^*) = w(y^*) - 8y^* \geq U_L(0) = w(y \neq y^*) \tag{35}$$

and

$$U_H(y^*) = w(y^*) - \frac{8y^*}{5} \geq U_H(0) = w(y \neq y^*). \tag{36}$$

The constraint on the Lows requires that $y^* \leq 0.25$ for a pooling equilibrium.

(10.4b) There is a continuum of separating equilibria, with different levels of y^*. What are the education levels and wages in the separating equilibria? Why are out-of-equilibrium beliefs needed, and what beliefs support the suggested equilibria? What are the self selection constraints for these with the out-of-equilibrium belief that $Pr(L|(y \neq y^*)) = 1$, and with $y_L = 0, y_H = y^*$. An out-of-equilibrium belief is needed because only $y = 0$ and $y = y^*$ occur in equilibrium, which leaves lots of other possibilities.

The self selection constraints say that High workers do not want to deviate by acquiring other than y^* of education (0 is most tempting), and the Lows do not want to deviate by acquiring y^* of education.

$$U_L(y^*) = w(y^*) - 8y^* \leq U_L(0) = w(y \neq y^*) \tag{38}$$

and

$$U_H(y^*) = w(y^*) - 8y^*/5 \geq U_H(0) = w(y \neq y^*). \tag{39}$$

These two constraints tell us that $y^* \geq 0.5$ and $y^* \leq 2.5$ in a separating equilibrium.

(10.4c) If you were forced to predict one equilibrium to be the one played out, which would it be?

Answer. The out-of-equilibrium beliefs are unsatisfactory in the pooling equilibria because acquiring more than $y = 0.5$ in education is a dominated strategy for the Low type. If one carries this reasoning further, only $y = 0.5$ is satisfactory, because separating equilibria with more signalling require the belief that $y = 0.5$ is a sign of a Low type.

CHAPTER 11

11.2: Selling Cars. A car dealer must pay \$10,000 to the manufacturer for each car he adds to his inventory. He faces three buyers. From the point of view of the dealer, Smith's valuation is uniformly distributed between \$11,000 and \$21,000, Jones's is between \$9,000 and \$11,000, and Brown's is between \$4,000 and \$12,000. The dealer's policy is to make a single take-it-or-leave-it offer to each customer, and he is smart enough to avoid making differerent offers to customers who could resell to each other. Use the notation that the maximum valuation is $\overline{V}$ and the range of valuations is R.

(11.2a) What will the offers be?

Answer. Let us use units of thousands of dollars. The expected profit from a customer with maximum valuation $\overline{V} > 10$ and range of valuations R is, if price P is charged:

$$\begin{aligned} \pi(P; V, R) &= \int_P^{\overline{V}} \frac{P-10}{R} dV \\ &= \left(\frac{PV}{R} - \frac{10V}{R}\right)\Big|_P^{\overline{V}} \\ &= \frac{\overline{V}P}{R} - \frac{10\overline{V}}{R} - \frac{P^2}{R} + \frac{10P}{R}. \end{aligned} \tag{40}$$

so

$$P^* = \frac{\overline{V}}{2} + 5. \tag{42}$$

Note that the optimal price does not depend on R, the range of possible valuations. Applying (42) to the specific customers: Smith will be offered $P = \frac{21}{2} + 5 = \$15,500$, Jones will be offered $P = \frac{11}{2} + 5 = \$10,500$, and Brown will be offered $P = \frac{12}{2} + 5 = \$11,000$. Moreover, Brown probably values the car less than Jones, but because of the higher probability that he values it more than \$10,000, he will end up paying more if he buys at all.

(11.2b) Who is most likely to buy a car? How does this compare with the outcome with perfect price discrimination under full information? How does it compare with the outcome when the dealer charges \$10,000 to each customer?

Answer. Smith will buy with probability 0.55, which is $\frac{21-15.5}{21-11}$. Jones will buy with probability 0.25. Brown will buy with probability 0.125. Thus, Smith is the buyer most likely to buy.

Whether the dealer charges \$10,000 or uses perfect price discrimination, the outcome is the same as far as allocative efficiency: Smith buys with probability 1, Jones buys with probability 0.5, and Brown buys with probability 0.25.

(11.2c) What happens to the equilibrium prices if with probability 0.25 each buyer has a valuation of \$0, but the probability distribution remains otherwise the same?

Answer. The prices are the same as in part (a). If a buyer values the car at less than \$10,000, it is irrelevant what his value may be, since it is unprofitable to sell to him anyway. Only the part of his distribution above \$10,000 matters to the seller's strategy. Note that this has the same flavor as the analysis of auctions, where a bidder's strategy is conditioned on his having the highest valuation, since if he does not, he will generally lose the auction anyway and his bid is irrelevant.

11.4: Incomplete Information.

(11.4a) What is the equilibrium in the game "Bargaining with Incomplete Information" if the probability of a low-valuation buyer is $\gamma = 0.1$, instead of 0.05 or 0.5?

Answer. In equilibrium, in the first period $p_1 = 150$, Buyer_{100} accepts $p_1 \leq 100$, and Buyer_{150} accepts p_1 with probability $m(p_1)$, where

$$\begin{cases} m = 1 & \text{if } p_1 \leq 105. \\ m = \alpha & \text{if } 105 < p_1 \leq 150, \\ & \quad (\text{where } 0 \leq \alpha \leq \frac{7}{9}) \\ m = 0 & \text{if } p_1 > 150. \end{cases}$$

In the second period, $p_2 = 150$, Buyer_{100} accepts $p_2 \leq 100$, and Buyer_{150}

second period from reducing the price to 100 and maintaining it at 150.

$$\pi(p_2 = 100) = 100 = \pi(p_2 = 150) = \frac{0.9(1-m^*)}{0.1+0.9(1-m^*)}(150). \qquad (43)$$

(11.4b) What level of γ marks the boundary between separating and pooling equilibria?

Answer. The boundary $\overline{\gamma}$ equals $\frac{1}{3}$. At that probability of a low-value buyer, even if it were known that zero high-valuation buyers had accepted the first-period offer, the seller would still not find it profitable to reduce his price to 100, because

$$\pi(p = 100) = 100 = \pi(p = 150) = \overline{\gamma}(0) + (1-\overline{\gamma})(150). \qquad (44)$$

11.6: A Fixed Bargaining Cost, Again. Apex and Brydox are entering into a joint venture that will yield 500 million dollars, but they must negotiate the split first. In bargaining round 1, Apex makes an offer at cost 0, proposing to keep A_1 for itself. Brydox either accepts (ending the game) or rejects. In Round 2, Brydox makes an offer at cost 10 million of A_2 for Apex, and Apex either accepts or rejects. In Round 3, Apex makes an offer of A_3 at cost c, and Brydox either accepts or rejects. If no offer is ever accepted, the joint venture is cancelled.

(11.6a) If $c = 0$, what is the equilibrium? What is the equilibrium outcome?

Answer. Apex: Offer $A_1 = 500$, accept $A_2 \geq 500$, and offer $A_3 = 500$. Brydox: Accept $A_1 \leq 500$, offer $A_2 = 500$, and accept $A_3 \leq 500$. Outcome: $A_1 = 500$, and it is accepted.

(11.6b) If $c = 10$, what is the equilibrium? What is the equilibrium outcome?

Answer. Apex: Offer $A_1 = 500$, accept $A_2 \geq 490$, and offer $A_3 = 500$. Brydox: Accept $A_1 \leq 500$, offer $A_2 = 490$, and accept $A_3 \leq 500$. Outcome: $A_1 = 500$, and it is accepted.

(11.6c) If $c = 300$, what is the equilibrium? What is the equilibrium outcome?

Answer. Apex: Offer $A_1 = 210$, accept $A_2 \geq 200$, and offer $A_3 = 500$. Brydox: Accept any $A_1 \leq 210$, offer $A_2 = 200$, and accept $A_3 \leq 500$. Outcome: $A_1 = 210$, and it is accepted.

CHAPTER 12

12.2: The Founding of Hong Kong. The Tai-Pan and Mr. Brock and the Tai-Pan loses 10. Moreover, Brock hates the Tai-Pan and receives 1 in utility for each 1 that the Tai-Pan pays out to get the land.

(12.2a) Fill in the entries in Table 12.2.

Table 12.2 "The Tai-Pan Game"

Winning bid:	1	2	3	4	5	6	7	8	9	10	11	12
If Brock wins:												
π_{Brock}												
$\pi_{Tai-Pan}$												
If Brock loses:												
π_{Brock}												
$\pi_{Tai-Pan}$												

Answer. See Table C.9.

Table C.9 "The Tai-Pan Game"

Winning bid:	1	2	3	4	5	6	7	8	9	10	11	12
If Brock wins:												
π_{Brock}	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6
$\pi_{Tai-Pan}$	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
If Brock loses:												
π_{Brock}	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{Tai-Pan}$	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10

(12.2b) In equilibrium, who wins, and at what bid?

Answer. There are two possible equilibrium outcomes: (i) The Tai-Pan wins with a bid of 10. The Tai-Pan bids 10 immediately, and is willing to bid up to 11. Brock is willing to bid up to 10. If Brock were to bid 11 and win, the Tai-Pan's payoff would be -10, which is no higher than if the Tai-Pan overbid to win at 12. (ii) The Tai-Pan wins with a bid of 11. The Tai-Pan bids 11 immediately, and is willing to bid up to 12. Brock is willing to bid up to 11. The outcome is that the Tai-Pan wins at 11, for a payoff of -9 (=-11 + 2). If the Tai-Pan were to deviate by bidding just 10, then Brock would bid 11 and the Tai-Pan would overbid to win at 12, for a payoff of $-10(=-12+2)$.

The essence of this problem is that winning is not the only reason to bid in an auction. The following reasoning is false: "The Tai-Pan will win with a bid of 3, because if Brock bids any higher, Brock will win and have a lower payoff than if he loses to the Tai-Pan's bid of 3." The reason this is false is that Brock does not intend to win when he bids higher than 3: he intends to lose anyway, but to harm the Tai-Pan more.

(12.2c) What happens if the Tai-Pan can precommit to a strategy?

Answer. The Tai-Pan can then precommit not to let Brock engage him in a malicious bidding war. The Tai-pan commits to pay no more than 3, and bids that immediately. Brock will not bid up to 4, because land?

Answer. The Tai-pan will bid up to 6. Brock bids up to 5. Thus, the Tai-Pan will bid 5 immediately, and win at that price.[9]

Problem 12.4: An Auction with Stupid Bidders. Smith's value for an object has a private component equal to 1 and component common with Jones and Brown. Jones's and Brown's private components both equal zero. Each player estimates the common component Z independently, and player i's estimate is either x_i above the true value or x_i below, with equal probability. Jones and Brown are naive and always bid their valuations. The auction is English.

(12.4a) If $x_{Smith} = 0$, what is Smith's dominant strategy if his estimate of Z equals 20?

Answer. Bid up to 21.

(12.4b) If $x_i = 8$ for all players and Smith estimates $Z = 20$, what are the probabilities that he puts on different values of Z?

Answer. $Prob(Z = 12) = Prob(Z = 28) = 0.5$. $Prob(Z \notin \{12, 28\}) = 0$.

(12.4c) If $x_i = 8$ but Smith knows that $Z = 12$ with certainty, what are the probabilities he puts on the different combinations of bids by Jones and Brown?

Answer. $Prob(4, 4) = Prob(4, 20) = Prob(20, 4) = Prob(20, 20) = 0.25$.

(12.4d) Why is 9 a better upper limit on bids for Smith than 21, if his estimate of Z is 20, and $x_i = 8$ for all three players?

Answer. First, consider bidding up to 9. With probability 0.5, the true value is just $Z = 12$, and in that case with probability 0.25 both Jones and Brown underestimate the value, and estimate it to equal 4. This is the only circumstance under which Smith could win with a bid as low as 9. He will win the auction at a price of 4, and earn a payoff of $1.125 (= 0.5 \cdot 0.25(13 - 4))$. Next, consider bidding up to 21. Again, with probability $0.5 \cdot 0.25$ the true value is $V = 12$ and Smith wins at a price of 4. But now with probability $0.5 \cdot 0.75$ the true value is $V = 12$ and Smith wins at a price of 20. Also, even if the true value is $V = 28$, which has probability 0.5, Smith wins if both Jones and Brown underestimate the value. Thus, Smith's payoff from the strategy of bidding up to 21 is $-0.375 (= 0.5 \cdot 0.25(13 - 4) + 0.5 \cdot 0.75(13 - 20) + 0.5 \cdot 0.25(29 - 20))$.

CHAPTER 13

13.2: Cournot Mergers. [10] There are three identical firms in an marginal cost is zero.

(13.2a) Compute the Cournot equilibrium price and quantities.

Answer.

$$\pi_1 = (1 - q_1 - q_2 - q_3)q_1 \tag{45}$$

Maximizing this gives $(1 - q_2 - q_3) - 2q_1 = 0$. In a symmetric equilibrium, $q_1 = q_2 = q_3$, so $1-2q-2q = 0$, and $q = 0.25$. The price is then $1-3q = 0.25$.

(13.2b) How do you know that there are no asymmetric Cournot equilibria, in which one firm produces a different amount than the others?

Answer. Each firm has a linear reaction curve, so the reaction curves cannot all intersect at more than one point unless the reaction curves are identical, which they certainly are not. We have shown that there exists one equilibrium which is symmetric, so there cannot be any other equilibrium.

Another way to see this is by using algebra. The first-order conditions are

$$\begin{aligned} 2q_1 + q_2 + q_3 &= 1 \\ q_1 + 2q_2 + q_3 &= 1 \\ q_1 + q_2 + 2q_3 &= 1 \end{aligned} \tag{46}$$

Subtracting the second of these equations from the first gives $q_1 - q_2 = 0$. Subtracting the third from the second gives $q_2 - q_3 = 0$. Hence, $q_1 = q_2 = q_3$ in equilibrium unless there are corner solutions, of which none exist here.

(13.2c) Show that if two of the firms merge, their shareholders are worse off.

Answer. The profit per firm from part (a) is 0.0625 (=(0.25)(0.25)). The sum of the profits of two firms is thus 0.125.

If the number of firms falls to two, we must solve the new Cournot game. Now, the first order condition is $(1 - q_2) - 2q_1 = 0$, so in a symmetric equilibrium $q = \frac{1}{3}$. The price is then $1 - 2q = \frac{1}{3}$. Profit per firm is about $0.111 (= (\frac{1}{3})(\frac{1}{3}) = \frac{1}{9})$. Thus, the merged firm has lower profits than its two component firms used to have.

13.4: Asymmetric Cournot Duopoly. Apex has variable costs of q_a^2 and a fixed cost of 1000, while Brydox has variables costs of $2q_b^2$ and no fixed cost. Demand is $p = 115 - q_a - q_b$.

(13.4a) What is the equation for Apex's Cournot reaction function?

Answer. This is found from the first order condition for Apex's maximization problem,

$$\underset{q_a}{Maximize} \; (115 - q_a - q_b)q_a - 1000 - q_a^2, \tag{47}$$

which yields $q_a = 28.75 - 0.25q_b$.

which yields $q_b = \frac{115}{6} - \frac{q_a}{6}$.

(13.4c) What are the outputs and profits in the Cournot equilibrium?

Answer. Solving the reaction functions together yields $q_a = 25$ and $q_b = 15$. The demand and cost curves can then be used to find that $\pi_a = 250$ and $\pi_b = 675$.