Answers to Even-Numbered Problems in Games and Information, Third Edition, Rasmusen

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This appendix contains answers to the even-numbered problems in the second edition of *Games and Information* by Eric Rasmusen, published in 1994.

CHAPTER 1

1.2: 2-by-2 Games. Find examples of 2-by-2 games with the following properties:

(1.2a) No Nash equilibrium (you can ignore mixed strategies).

Answer. See "Simple Cycling" (Table A.1).

Table A.1 "Simple Cycling"

		Jones		
		Left		*Right*
	Up	1,0	→	0, 1
Smith:		↑		↓
	Down	0, 1	←	1,0

Payoffs to: (Smith, Jones).

(1.2b) No weakly pareto-dominant strategy combination.

Answer. See "Simple Cycling" (Table A.1).

(1.2c) At least two Nash equilibria, including one equilibrium that pareto-dominates all other strategy combinations.

Answer. In "Ranked Coordination" (Table 1.7). *(Large, Large)* has uniformly higher payoffs than *(Small, Small).*

(1.2d) At least three Nash equilibria.

<u>*Answer.*</u> In "Everything an Equilibrium" (Table 1.3), every strategy profile is a Nash equilibrium.

1.4: Discoordination. Suppose that a man and a woman each choose whether to go to a prize fight or a ballet. The man would rather go to the prize fight, and the woman to the ballet. What is more important to them, however, is that the man wants to show up to the same event as the woman, but the woman wants to avoid him.

(1.4a) Construct a game matrix to illustrate this game, choosing numbers to fit the preferences described verbally.

<u>*Answer.*</u> See "The Battle of the Sexes with Unrequited Love" (Table A.4).

Table A.4 "The Battle of the Sexes with Unrequited Love"[1]

		Woman		
		Prize Fight		*Ballet*
	Prize Fight	$20, -2$	$\rightarrow$	$-10, 2$
Man		$\uparrow$		$\downarrow$
	Ballet	$-20, 1$	$\leftarrow$	$10, -1$

Payoffs to: (Man, Woman)

(1.4b) If the woman moves first, what will happen?

<u>*Answer.*</u> (*Ballet*, *Ballet*).

(1.4c) Does the game have a first-mover advantage?

<u>*Answer.*</u> No— it has a first-mover disadvantage.

(1.4d) Show that there is no Nash equilibrium if the players move simultaneously.

<u>*Answer.*</u> (*Prize Fight*, *Ballet*) and (*Ballet*, *Prize Fight*) are not Nash because the man would deviate; (*Prize Fight*, *Prize Fight*) and (*Ballet*, *Ballet*) are not, because the woman would.[2]

CHAPTER 2

2.2: Elmer's Apple Pie. Mrs Jones has made an apple pie for her son, Elmer, and she is trying to figure out whether the pie tasted divine, or

ravenous, or merely hungry, and he will eat either 2, 3, or 4 pieces of pie. Mrs Jones knows he is ravenous half the time (but not which half). If the pie is divine, then, if Elmer is hungry, the probabilities of the three consumptions are (0, 0.6, 0.4), but if he is ravenous the probabilities are (0,0,1). If the pie is just good, then the probabilities are (0.2, 0.4, 0.4) if he is hungry and (0.1,0.3,0.6) if he is ravenous.
Elmer is a sensitive, but useless, boy. He will always say that the pie is divine and his appetite weak, regardless of his true inner feelings.

(2.2a) What is the probability that he will eat four pieces of pie?

Answer. P(4) = 17/30 (about 0.57) $(= P(4|Divine)P(Divine) + P(4|Good)P(Good) = ((\frac{1}{2}) \cdot 0.4 + (\frac{1}{2}) \cdot 1)(\frac{1}{3}) + ((\frac{1}{2}) \cdot 0.4 + (\frac{1}{2}) \cdot 0.6))(\frac{2}{3})$.

(2.2b) If Mrs Jones sees Elmer eat four pieces of pie, what is the probability that he is ravenous and the pie is merely good?

Answer. $P(Ravenous, Good|4) = 6/17$ (about 0.35) $(= \frac{P(4|RG)P(RG)}{P(4)} = \frac{0.6((\frac{1}{2})\cdot(\frac{2}{3}))}{(17/30)})$.

(2.2c) If Mrs Jones sees Elmer eat four pieces of pie, what is the probability that the pie is divine?

Answer. $P(Divine|4) = 7/17$ (about 0.41) $(= \frac{P(4|D)P(D)}{P(4)} = \frac{((\frac{1}{2})\cdot 0.4+(\frac{1}{2})\cdot 1)(\frac{1}{3})}{17/30})$.

2.4: The Battleship Problem.[3] The Pentagon has the choice of building one battleship or two cruisers. One battleship costs the same as two cruisers, but a cruiser is sufficient to carry out the navy's mission— if the cruiser survives to get close enough to the target. The battleship has a probability of p of carrying out its mission, whereas a cruiser only has probability $p/2$. Whatever the outcome, the war ends and any surviving ships are scrapped. Which option is superior?

Answer. The battleship completes its mission with probability p. Each cruiser is sunk with probability $1 - p/2$, so both are sunk with probability $(1-p/2)^2$, and then the mission fails. Hence, at least one cruiser survives to complete the mission with probability $1-(1-p/2)^2$, which equals $p-(p^2/4)$, which is less than p. Therefore, buy the battleship.

2.6: California Drought. California is in a drought and the reservoirs are running low. The probability of rainfall in 1991 is 1/2, but with probability 1 there will be heavy rainfall in 1992. The state uses rationing rather than the price system, and it must decide how much water to consume in 1990 and how much to save till 1991. Each Californian has a utility function of $U = log(w_{90}) + log(w_{91})$. Show that the state should allocate twice as much water to 1990 as to 1991.[4]

it rains in 1991 the saved water won't be needed. Differentiating with respect to w_{90} and equating to zero gives $w_{90}^{*} = (\frac{2}{3})\overline{w}$. Thus, two thirds of the water should be consumed the first year.

CHAPTER 3

3.2: Running from the Gestapo. Two risk-neutral men, Schmidt and Braun, are walking south along a street in Nazi Germany when they see a single Gestapo agent coming to check their papers. Only Braun has his papers (unknown to the Gestapo, of course). The Gestapo agent will catch both men if both or neither of them run north, but if just one runs, he must choose which one to stop— the walker or the runner. The penalty for being without papers is 24 months in prison. The penalty for running away from an agent of the state is 24 months in prison, on top of the sentences for any other charges, but the conviction rate for this offense is only twenty-five percent. The two friends want to maximize their joint welfare, which the Gestapo man wants to minimize. Braun moves first, then Schmidt, then the Gestapo.

(3.2a) What is the outcome matrix for outcomes that might be observed in equilibrium? (Use θ for the probability that the Gestapo chases the runner and γ for the probability that Braun runs.)

Answer. We can rule out both Schmidt and Braun either walking or running, by iterated dominance. The first strategy the Gestapo would eliminate is: "Stand still while both men go away together." Once that is eliminated, Schmidt and Braun find it weakly dominant to split their actions. If Braun runs, Schmidt will walk; if Braun walks, Schmidt will run. Otherwise, the Gestapo man will catch both. The expected penalty from running and being caught, aside from any penalty for not having papers, is 6 months in prison (=(1/4) 24). The payoffs to the friends from the various remaining outcomes are shown in Table C.4.

Table C.4 Running from the Gestapo

	Gestapo chases runner (θ)	Gestapo chases walker ($1-\theta$)
Braun runs (γ)	-6	-24
Schmidt runs ($1-\gamma$)	-30	0

(3.2b) What is the probability that the Gestapo agent chases the runner, (call it θ^*)?

Answer. There is no pure strategy equilibrium. In the mixed-strategy equilibrium, the friends' payoffs must be equal from the two pure strategies, $\pi(Braun\ runs) = \pi(Schmidt\ runs)$, so

(3.2c) What is the probability that Braun runs, (call it γ^*)?

Answer. In the mixed-strategy equilibrium, the Gestapo payoffs must be equal from the two pure strategies: $\pi(chase\ runner) = \pi(chase\ walker)$, so

$$\gamma(6) + (1-\gamma)(30) = \gamma(24) + (1-\gamma)(0). \tag{2}$$

Therefore, $\gamma^* = \frac{5}{8}$.

(3.2d) Since Schmidt and Braun share the same objectives, is this a cooperative game?

Answer. No. Their utility function is the same, and the model is set up so that they can take the same action without coordination difficulties, but the modeller is not trying to discover what sort of binding agreement they might make with the Gestapo.[5]

A Goethe University PhD-student group pointed out to me that this is actually a game of incomplete information, a topic not yet covered by Chapter 3 of the book. The Gestapo man does not know the payoffs of the game, because he does not know whether his payoff is higher from chasing Schmidt or Braun. They know, so it is also a game of incomplete information.

We do not need the special techniques of incomplete information, though, because this game can also be modelled as a simultaneous game of complete information. Think of the actions as (Gestapo: "Chase the Runner" or "Chase the Walker") (S-B: "Man with Papers Runs", "Man without Papers Runs").

3.4: Mixed Strategies in the Battle of the Sexes. Refer back to the Battle of the Sexes and Ranked Coordination in Section 1.4. Denote the probabilities that the man and woman pick *Prize Fight* by γ and θ.

(3.4a) Find an expression for the man's expected payoff.

Answer. $\pi_m = \gamma(2\theta + (0)[1-\theta]) + (1-\gamma)((0)\theta + 1[1-\theta])$.

(3.4b) Find the first order condition for the man's choice of strategy.

Answer. $\frac{d\pi_m}{d\gamma} = 2\theta - [1-\theta]) = 0$.

(3.4c) What are the equilibrium values of γ and θ, and the expected payoffs?

Answer. $\theta^* = 1/3, \gamma^* = 2/3$, $\pi_m = \frac{2}{3}, \pi_w = \frac{2}{3}$.

(3.4d) Find the most likely outcome and its probability.

Answer. *(Prize Fight, Ballet)* for (M, W), which has probability 4/9, about 0.444.

[5] This models a joke that continues like this: Braun was more than a little overweight

(3.4e) What is the equilibrium payoff in the mixed- strategy equilibrium for Ranked Coordination ?

Answer. The probability is found by solving $2\theta+(1-\theta)(-1) = (-1)\theta+(1-\theta)$. Therefore, $Prob(Large) = 2/5$, $\pi = 1/5$ for each player.

(3.4f) Why is the mixed-strategy equilibrium a better focal point in the Battle of the Sexes than in Ranked Coordination?

Answer. "Ranked Coordination" has a Pareto-dominant Nash equilibrium, so if players are optimistic, they will focus on that equilibrium. In "The Battle of the Sexes", neither Nash equilibrium is pareto-dominant.

CHAPTER 4

4.2. Curly, Moe, and Larry.[6] Three salesmen, Curly, Moe, and Larry, are trying to sell electrical generators to a large customer. Price is not the focus: rather, it is whether service will be included in the contract or not. Each seller would profit by 5 if service is not included, and 1 if service is included. If all the offers are equal, Curly wins the order with probability 0.5, Moe with probability 0.4, and Larry with probability 0.1. If some offers include service and others do not, however, the offers that do not include service are rejected for sure. If Moe and Curly offer service, and Larry does not, for example, then Larry's probability falls to zero, Curly's rises to 0.55, and Moe's rises to 0.45.

The offers are made in sequence, and publicly. First Larry offers, then Moe, and then Curly. What happens?

xxx **4.4: Perfect Blackmail.** Politician Smith has pictures of politician

Jones dressed in frilly underwear, while Jones has tapes of Smith promising a woman a government job in exchange for her favors. The harm from public exposure is 20 for Smith and 50 for Jones. Smith threatens to show the pictures of Jones if Jones votes against a tariff. Smith receives an extra utility of 3 if Jones votes for the tariff, but Jones loses 4 in utility from that vote. Smith would get utility of 5 from showing the pictures of Jones, and Jones would get utility of 7 from playing the tapes of Smith. The vote is one month from today, but you may assume that both politicians live forever.

(4.4a) What is a perfect equilibrium for this game?

ANSWER: One answer is: *Smith shows the pictures. Jones votes against the tariff and plays the tapes.* Given that Jones's behavior is unconditional, Smith should show the pictures to get the extra utility of 5 from that action. Given that Smith's behavior is unconditional, (4.4b) What is an equilibrium for this game that is Nash but not perfect?

ANSWER: One answer is: *Smith shows the pictures iff Jones votes against the tariff or Jones plays the tapes first. Jones votes for the tariff and plays the tapes iff Smith shows the pictures first.* The equilibrium OUTCOME (distinct from the equilibrium, which is a STRATEGY COMBINATION) is that Jones votes for the tariff and nobody exposes anybody.

This is Nash. If Smith deviates and shows the pictures of Jones, Jones will retaliate by exposing him, for a net loss of 15 (=20-5) to Smith. If Jones deviates and plays the tape of Smith, Smith will retaliate by showing the pictures, for a net loss of 43 (=50-7) for Jones. If Jones deviates and votes against the tariff, both politicians will will have their secrets exposed, and Jones will have a net loss of 39 (=50-7-4).

This is not perfect. Suppose we start the game with Jones having deviated by voting against the tariff. If Smith follows his assigned strategy of showing the pictures, Jones will play the tapes in retaliation, so Smith will suffer a net loss of 15 (=20-5). Thus, Smith should deviate by not showing the pictures if Jones votes against the tariff.

In this game, the moves do not follow a neat sequence. The first move is clearly that Jones votes for or against the tariff, but thereafter, either player has the option to expose the other's secrets at any time. Thus, we need to consider both Smith showing the pictures first and Jones playing the tapes first. Both of them have a threat available, but know that there is a counterthreat.

PROBLEMS FOR CHAPTER 5 Reputation and Repeated Games

5.2: Product Quality with Lawsuits. Modify the Product Quality game of Section 5.4 by assuming that if the seller misrepresents his quality he must, as a result of a class-action suit, pay damages of x per unit sold, where $x \in (0, c]$ and the seller becomes liable for x at the time of sale.

(5.2a) What is $\tilde{p}$ as a function of $x, F, c,$ and r? Is $\tilde{p}$ greater than when $x = 0$?

Answer. To avoid low quality, it must be that $\frac{q(p-x)}{1+r} = \frac{q(p-c)}{r}$,and so $rqp - rqx = qp - qc + rqp - rqc$, and $0 = rx + p - c - rc$, and $\tilde{p} = (1+r)c - rx$. Thus, the threat of a lawsuit reduces the quality-ensuring price.

(5.2b) What is the equilibrium output per firm? Is it greater than when $x = 0$?

Answer. $\frac{q_i((1+r)c-rx-c)}{r} = F$, for zero profits ex ante. Thus, $q_i(c + rc - rx - c) = rF$, and $q_i = \frac{F}{c-x}$. The equilibrium output has increased because $x > 0$.

Answer. For supply to equal demand, $\frac{nF}{c-x} = q(p)$, so that

$$\tilde{n} = \frac{(c-x)q(\tilde{p})}{F}. \qquad (3)$$

This might either increase or decrease in x, because $(c-x)$ decreases in x, but $\tilde{p}$ decreases also, so $q(\tilde{p})$ increases when $x > 0$.

(5.2d) If, instead of x per unit, the seller pays X to a law firm to successfully defend him, what is the incentive compatibility constraint?

Answer. The incentive compatibility constraint, $\pi_{low} \leq \pi_{high}$, becomes

$$\frac{q_i p}{1+r} - \frac{X}{1+r} \leq \frac{q_i(p-c)}{r}. \qquad (4)$$

The question does not ask for a solution, but this equation yields

$$p = (1+r)c - \frac{rX}{q_i}, \qquad (5)$$

which must be solved simultaneously with the other equations to find p and q_i.

5.4: Repeated Entry Deterrence. Assume that Entry Deterrence I is repeated an infinite number of times, with a tiny discount rate and with payoffs received at the start of each period. In each period, the entrant chooses *Enter* or *Stay out*, even if he entered previously.

(5.4a) What is a perfect equilibrium in which the entrant enters each period?

Answer. $(Enter, Collude)$ each period.

(5.4b) Why is (*Stay out, Fight*) not a perfect equilibrium?

Answer. $(Stay\ out, Fight|Enter)$ gives the incumbent no incentive to choose $Fight$. Given the entrant's strategy, if somehow the game ends up off the equilibrium path with the entrant having entered, the entrant will *Stay Out* in succeeding periods. Hence, the incumbent would deviate by choosing $Collude$ and getting 50 instead of 0.

(5.4c) What is a perfect equilibrium in which the entrant never enters?

Answer. Entrant: *Stay out* unless the incumbent has chosen $Collude$ in some previous period, in which case, $Enter$.

Incumbent: $Fight|Enter$ unless the incumbent has chosen $Collude$ in some previous period, in which case, choose $Collude|Enter$.

In this equilibrium, the incumbent suffers a heavy penalty if he ever colludes.

and the incumbent has never yet colluded. The incumbent's choice is between

$$\pi(collude) = 50 + \frac{50}{r} \tag{6}$$

and

$$\pi(fight) = 0 + \frac{100}{r} \tag{7}$$

These two payoffs equal each other if $r = 2$, so if the discount rate is anything less, the equilibrium in (c) remains an equilibrium.

5.6: Evolutionarily Stable Strategies. A population of scholars are playing the following coordination game over their two possible conversation topics over lunch, football and economics. Let $N_t(F)$ and $N_t(E)$ be the numbers who talk football and economics in period t, and let θ be the percentage who talk football, so $\theta = \frac{N(football)}{N(football)+N(economics)}$. Government regulations requiring lunchtime attendance and stipulating the topics of conversation have maintained the values $\theta = 0.5$, $N_t(F) = 50,000$ and $N_t(E) = 50,000$ up to this year's deregulatory reform. In the future, some people may decide to go home for lunch instead, or change their conversation. Table 5.8 shows the payoffs.

Table 5.8 Evolutionarily Stable Strategies

		Scholar 2	
		Football (θ)	*Economics* ($1-\theta$)
	Football (θ)	1,1	0, 0
Scholar 1			
	Economics ($1-\theta$)	0, 0	5,5

Payoffs to: (Scholar 1, Scholar 2).

(5.6a) There are three Nash equilibria: *(Football, Football)*, *(Economics, Economics)*, and a mixed-strategy equilibrium. What are the evolutionarily stable strategies?

Answer. *Football* and *Economics*. The mixed strategy can be invaded by either of these, since each strategy does very well against itself, and also does well in the mixed population.

(5.6b) Let $N_t(s)$ be the number of scholars playing a particular strategy in period t and let $\pi_t(s)$ be the payoff. Devise a Markov difference equation to express the population dynamics from period to period: $N_{t+1}(s) = f(N_t(s), \pi_t(s))$. Start the system with a population of 100,000, half the scholars talking football and half talking economics. Use your dynamics to finish Table 5.9.

Table 5.9 Conversation Dynamics

t	$N_t(F)$	$N_t(E)$	θ	$\pi_t(F)$	$\pi_t(E)$

Answer. One of the many possible dynamics is $N_{t+1}(s) = N_t(s)[0.4\pi_t(s)]$. This is shown in Table C.5.

Table C.5 Markov Conversation Dynamics

t	$N_t(F)$	$N_t(E)$	θ	$\pi_t(F)$	$\pi_t(E)$
-1	50,000	50,000	0.5	0.5	2.5
0	50,000	50,000	0.5	0.5	2.5
1	10,000	50,000	0.167	0.167	4.167
2	667	83,333	0.007	0.007	4.965

1. (5.6c)] Repeat part (b), but specifying non-Markov dynamics, in which $N_{t+1}(s) = f(N_t(s), \pi_t(s), \pi_{t-1}(s))$.

   Answer. $N_{t+1}(s) = N_t(s)[0.2\pi_t(s)+0.2\pi_{t-1}(s)]$. This is shown in Table C.6.

Table C.6 Non-Markov Conversation Dynamics

t	$N_t(F)$	$N_t(E)$	θ	$\pi_t(F)$	$\pi_t(E)$
-1	50,000	50,000	0.5	0.5	2.5
0	50,000	50,000	0.5	0.5	2.5
1	10,000	50,000	0.167	0.167	2.5
2	1,333	66,667	0.020	0.02	4.90

PROBLEMS FOR CHAPTER 6 Dynamic Games with Asymmetric Information

Problem 6.2: Limit Pricing.[7] An incumbent firm operates in the local computer market, which is a natural monopoly in which only one firm can survive. The incumbent can price *Low*, losing 40 in profits, or *High*, losing nothing. It knows its own operating cost C, which is 20 with probability 0.75 and 30 with probability 0.25. A potential entrant knows those probabilities, but not the incumbent's exact cost. The entrant can enter at a cost of 100, and its operating cost of 25 is common knowledge. The firm with the highest operating cost immediately drops out if it has a competitor, and the survivor earns the monopoly revenue of 150.

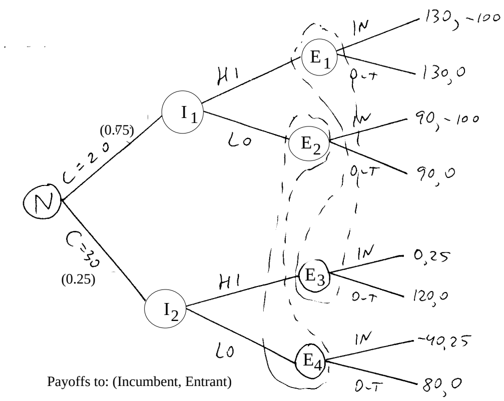
(6.2a) Draw the extensive form for this game.

Answer. See Figure C.1.

Figure C.1 The Extensive Form for "Limit Pricing"

[7] See Milgrom and Roberts (1982a).

Figure A1.7: The Extensive Form for ''Limit Pricing''



(6.2b) Why is there no perfect equilibrium in which the incumbent prices *Low* only when its costs are 20? (no separating equilibrium)

<u>*Answer.*</u> In any such separating equilibrium, the entrant would not enter if the incumbent priced *Low*. Therefore, the incumbent would price *Low* in the first period even if $C = 30$, for a payoff of 80 instead of the 0 payoff from pricing $High$ and inducing entry. This strategy of $Low|30$ contradicts the postulated separating equilibrium. Note also that if it cost 125 to price *Low*, a separating equilibrium would indeed exist, because the incumbent would be unwilling to pay 125 to fool the entrant into thinking that $C = 20$ when really $C = 30$.

(6.2c) In a perfect Bayesian equilibrium in which the incumbent prices *Low* regardless of its costs (a pooling equilibrium), what do out-of-equilibrium beliefs have to be specified?

<u>*Answer.*</u> Beliefs must be specified for the probability that $C = 20$ given that the incumbent prices $High$.

(6.2d) What are two different perfect Bayesian equilibria for this game?

<u>*Answer.*</u> From (b), both equilibria must be pooling.

$$(i)\{(Low|20, Low|30), Out|Low, In|High, Prob(20|High) = 0\}$$
$$(ii)\{(High|20, High|30), Out|Low, Out|High, Prob(20|Low) = 0.5\}.$$

(6.2e) What is a set of out-of-equilibrium beliefs that do not support a pooling equilibrium at a *Low* price?

<u>*Answer.*</u> $Prob(20|High) = 1$. If this is the belief, then the incumbent with a cost of 20 would deviate to a $High$ price, increasing his profits without inducing entry.

(6.4a) Why does $Pr(Strong|Enter, Nature\ said\ nothing) = 0.95$ not support the equilibrium in Section 6.3?[8]

Answer. Under these beliefs, if the entrant deviates and enters, the incumbent's expected payoff from *Fight* is 15 $(= 0.95(0) + 0.05(300))$, which is less than the 50 he can get from *Collude*.

(6.4b) Why is the equilibrium in Section 6.3 not an equilibrium if 0.7 is the probability that Nature tells the incumbent?

Answer. The entrant would deviate to $Enter|Strong$. If the entrant is strong, he expects the incumbent to fight with probability 0.3 and collude with probability 0.7. The payoff from entry is then 25 $(= 0.3(-10) + 0.7(40))$, which is greater than the 0 from staying out.

(6.4c) Describe the equilibrium if 0.7 is the probability that Nature tells the incumbent. For what out-of-equilibrium beliefs does this remain the equilibrium?

Answer. The equilibrium when Nature tells with probability 0.7 is in mixed strategies, because in a pure-strategy equilibrium the incumbent could deduce the entrant's type from whether the entrant enters or not. If only strong entrants entered, the incumbent would never fight entry, and weak entrants would also enter. The equilibrium is

Entrant: $Enter|Strong$
$Enter\ with\ probability\ \theta = 0.2|Weak$

Incumbent: *Collude*| *(Enter, Nature said "Strong")*
Fight—(Enter, Nature said "Weak"),
$Collude\ with\ probability\ \gamma = 17/22|(Enter, Nature\ said\ nothing)$

The strong entrant enters because his expected payoff is

$$\begin{aligned} \pi_e(Enter|Strong) &= 0.7(40) + 0.3(\gamma(40) + (1-\gamma)(-10)) \\ &= 28 + 12(17/22) - 3(5/22) \\ &> 0. \end{aligned} \tag{8}$$

The weak entrant must be indifferent between entering and staying out, so

$$\pi_e(Enter|Weak) = 0.7(-10) + 0.3(\gamma(40) + (1-\gamma)(-10)) = \pi_e(Stay\ out|Weak) = 0, \tag{9}$$

which when solved yields $\gamma = 17/22$.

If the incumbent observes that the entrant has entered, he knows that the entrant might be either strong (probability 0.5) or weak (probability 0.5θ). Using Bayes's Rule and equating the incumbent's payoffs from fighting and colluding gives

Solving equation (12) yields $\theta = 0.2$.

Since there is no behavior that could never be observed in in equilibrium, no out-of-equilibrium beliefs need be specified.

PROBLEMS FOR CHAPTER 7 Moral Hazard: Hidden Actions

7.2: The Principal-Agent Problem. Suppose the agent has a utility function of $U = \sqrt{w} - e$, where e can assume the levels 0 or 7, and a reservation utility of $\overline{U} = 4$. The principal is risk neutral. Denote the agent's wage, conditioned on output, as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 1,000. Only the agent observes his effort. Principals compete for agents. Table 7.6 shows the output.

Table 7.6 Output from Low and High Effort

	Probability of Output of		
Effort	0	1000	Total
Low $(e = 0)$	0.9	0.1	1
High $(e = 7)$	0.2	0.8	1

(7.2a) What are the incentive compatibility, participation, and zero-profit constraints for obtaining high effort?

<u>*Answer*</u>. The incentive compatibility constraint is

$$0.2\sqrt{\underline{w}} + 0.8\sqrt{\overline{w}} - 7 \geq 0.9\sqrt{\underline{w}} + 0.1\sqrt{\overline{w}}. \tag{11}$$

The participation constraint is

$$0.2\sqrt{\underline{w}} + 0.8\sqrt{\overline{w}} - 7 \geq 4. \tag{12}$$

The zero-profit constraint is

$$0.2\underline{w} + 0.8\overline{w} = 0.2(0) + 0.8(1000) = 800. \tag{13}$$

(7.2b) What would utility be if the wage were fixed and could not depend on output or effort?

<u>*Answer*</u>. Effort would be low, so the expected value of output would be 100 $(= 0.9(0) + 0.1(1000))$. The wage would be a flat 100, and utility would be 10 $(= \sqrt{100} - 0)$.

(7.2c) What is the optimal contract? What is the agent's utility?

The zero-profit constraint is also binding, and can be written as:

$$\underline{w} = 4000 - 4\overline{w}. \tag{15}$$

Substituting the zero-profit constraint into the incentive compatibility constraint yields

$$\sqrt{\overline{w}} - \sqrt{4000 - 4\overline{w}} = 10. \tag{16}$$

If $\overline{w} = 900$ and $\underline{w} = 400$, then since $30 - \sqrt{4000 - 3600} = 10$, we have a solution. The agent's utility is 21, from

$$0.2\sqrt{400} + 0.8\sqrt{900} - 7 = 21. \tag{17}$$

(7.2d) What would the agent's utility be under full information? Under asymmetric information, what is the agency cost (the lost utility) as a percentage of the utility the agent receives?

Answer. Under full information, the agent would be perfectly insured and would choose high effort. To see that he would choose high effort, note that his wage would be 100 and his utility would be 10 ($= \sqrt{100} - 0$) under low effort. Under high effort, his wage would be 800 ($= 0.2(0) + 0.8(1000)$), and his utility would be 21.3 ($= \sqrt{800} - 7$). Since the utility under asymmetric information is 21, and the difference is 0.3, the agency cost is 0.3/21, about 1.4 percent .

7.4: Authority. A salesman must decide how hard to work on his own time on getting to know a potential customer. If he exerts effort X, he incurs a utility cost $X^2/2$. With probability X, he can then go to customer X and add V to his own earnings. With probability (1-X), he offends the customer, and on going to him would subtract L from his earnings. The boss will receive benefit B from the sale in either case. The ranking of these numbers is $V > L > B > 0$. The boss and the salesman have equal bargaining power, and are free to make side payments to each other.

(7.4a) What is the first-best level of effort, X_a?

Answer. Total surplus is

$$-\frac{X^2}{2} + X(V + B),$$

so

$$-X + (V + B) = 0,$$

and

$$X_a = V + B$$

Total surplus is

$$-\frac{X^2}{2} + X(V + B),$$

(7.4b) If the boss has the authority to block the salesman from selling to this customer, but cannot force him to sell, what value will X take?

Answer. If the salesman is successful, the total benefit from the sale will be $V+B$, split between boss and salesman. The salesman therefore maximizes

$$-\frac{X^2}{2} + X\frac{V+B}{2},$$

so

$$-X + \frac{V+B}{2} = 0,$$

and

$$X = \frac{V+B}{2}.$$

(7.4c) If the salesman has the authority over the decision on whether to sell to this customer, and can bargain for higher pay, what will his effort be?

Answer. If the salesman is successful, he will want to make the sale to get V for himself. He cannot bargain for more from his boss, because the boss knows the salesman will make the sale even if agreement is not reached; any threat not to make the sale is not credible. Thus, the salesman's payoff is therefore

$$-\frac{X^2}{2} + XV,$$

which when maximized yields

$$X = V.$$

(7.4d) Rank the effort levels X_a, X_b, and X_c in the previous three sections.

Answer. $X_a > X_c > X_b$.

PROBLEMS FOR CHAPTER 8 Further Topics in Moral Hazard

8.2: Monitoring with Error: Second Offenses.[9] Individuals who are risk-neutral must decide whether to commit zero, one, or two robberies. The cost to society of robbery is 10, and the benefit to the robber is 5. No robber is ever convicted and jailed, but the police beat up any suspected robber they find. They beat up innocent people mistakenly sometimes, as shown by Table 8.2, which shows the probabilities of zero or more beatings for someone who commits zero, one, or two robberies.

Table 8.2 Crime

	Beatings		
Robberies	0	1	2

(8.2a) How big should p^*, the disutility of a beating, be made to deter crime completely while inflicting a minimum of punishment on the innocent?

Answer. If $\pi(n)$ is the payoff from committing n robberies, the payoffs are

$$\begin{aligned} \pi(0) &= -0.18p^* - 0.01(2p^*) = -0.2p^* \\ \pi(1) &= 5 - 0.34p^* - 0.06(2p^*) = 5 - 0.46p^* \\ \pi(2) &= 10 - 0.42p^* - 0.09(2p^*) = 10 - 0.6p^* \end{aligned} \quad (18)$$

Incentive compatibility requires that $\pi(0) \geq \pi(1)$ and $\pi(0) \geq \pi(2)$, so $-0.2p^* \geq 10 - 0.6p^*$ and $-0.2p^* \geq 5 - 0.46p^*$. These are solved by $p^* \geq 19.2$ (rounded) and $p^* \geq 25$, so since both inequalities must be true, $p^* = 25$.

Note that the problem narrows the issue to policies which completely deter robbery. The optimal level of robbery is a different issue.

(8.2b) In equilibrium, what percentage of beatings are of innocent people? What is the payoff of an innocent man?

Answer. 100 percent of the beatings are of innocent people. The payoff of an innocent person is -5 ($= \pi(0) = -0.2(25)$).

(8.2c) Now consider a more flexible policy, which inflicts heavier beatings on repeat offenders. If such flexibility is possible, what are the optimal severities for first- and second-time offenders? (call these p_1 and p_2). What is the expected utility of an innocent person under this policy?

Answer. One of the incentive compatibility constraints is that $\pi(0) \geq \pi(1)$, so $-0.18p_1 - 0.01(p_1+p_2) \geq 5 - 0.34p_1 - 0.06(p_1+p_2)$. The problem is to maximize $-0.18p_1 - 0.01(p_1+p_2)$ subject to that constraint, which can be rewritten as $0.21p_1 + 0.05p_2 \geq 5$. Since the maximand is linear in the control variables, the solution will be a corner solution with a binding constraint. Increasing p_1 by one unit subtracts 0.19 from the maximand while adding 0.21 to the constraint, a ratio of about -0.9, while increasing p_2 yields a ratio of -0.2, so $p_1 = 0$ in the solution. Solving the problem subject to the incentive compatibility constraint that $\pi(0) \geq \pi(2)$ also would give $p_1 = 0$ because there, too, the best way for the government to put a wedge between the payoff of the innocent and the guilty is by focussing on second offenses.

Since $p_1 = 0$, the payoffs are

$$\begin{aligned} \pi(0) &= -0.01p_2, \\ \pi(1) &= 5 - 0.06p_2, \\ \pi(2) &= 10 - 0.09p_2. \end{aligned} \quad (19)$$

Again, committing two robberies is the most tempting deviation, so the binding incentive compatibility constraint is $-0.01p_1 = 10 - 0.09p_2$, which yields $p_1 = 125$, a severe beating indeed. The payoff of an innocent man is -1.25 ($= \pi(0) = -0.01(125)$), higher than with penalty

Table 8.3 More Crime

	Beatings		
Robberies	0	1	2
0	0.9	0.1	0
1	0.6	0.3	0.1
2	0.5	0.3	0.2

Answer. Since there is no chance of an innocent person being falsely accused twice, the optimal penalty is 0 for the first offense and 1,000 (or some other large number) for the second offense. In this case, the limitation of the penalty for first-time offenders to 0 is not a constraint; it arises at the unconstrained optimum.

8.4: Teams. A team of two workers produces and sells widgets for the principal. Each worker chooses high or low effort. An agent's utility is $U = w - 20$ if his effort is high, and $U = w$ if it is low, with a reservation utility of $\overline{U} = 0$. Nature chooses business conditions to be excellent, good, or bad, with probabilities θ_1, θ_2, and θ_3. The principal observes output but not business conditions, as shown in Table 8.5.

Table 8.5: Team Output

	Excellent (θ_1)	**Good** (θ_2)	**Bad** (θ_3)
High, High	100	100	60
High, Low	100	50	20
Low, Low	50	20	0

(8.4a) Suppose $\theta_1 = \theta_2 = \theta_3$. Why is $\{(w(100) = 30, w(not\ 100) = 0), (High, High)\}$ not an equilibrium?

Answer. A worker would deviate. His payoff from $High$ is $\pi(High) = \frac{2}{3}(30) - 20 = 0$, and his payoff from Low is $\pi(Low) = \frac{1}{3}(30) = 10 > 0$.

(8.4b) Suppose $\theta_1 = \theta_2 = \theta_3$. Is it optimal to induce high effort? What is an optimal contract with nonnegative wages?

Answer. High effort by both workers is efficient here. The expected output minus the real cost of labor for HH is $46\frac{2}{3}$ $(= \frac{2}{3}(100) + \frac{1}{3}(60) - 40)$; from HL it is $36\frac{2}{3}$ $(= \frac{1}{3}(100) + \frac{1}{3}(50) + \frac{1}{3}(20) - 20)$; from LL it is $23\frac{1}{3}$ $(= \frac{1}{3}(50) + \frac{1}{3}(20))$. An optimal contract is the boiling-in-oil contract $((w = 60|(q = 60), w = 0|(q \neq 60))$. This satisfies the participation constraint by giving each worker an expected utility of 0 $(= \frac{1}{3}(60) - 20)$, and the incentive compatibility constraint by making a worker's expected utility 0 if he chooses low effort.

$(= \frac{1}{2}(100 + 50) - 20$; from LL it is 35 $(= \frac{1}{2}(50 + 20))$. An optimal contract is $(w = -3000|(q = 50), w = 20|(q \neq 50))$. This satisfies the participation constraint by giving each worker an expected utility of 0 $(= 20 - 20)$, and the incentive compatibility constraint by making a worker's expected utility very negative if he chooses low effort.

(8.4d) Should the principal stop the agents from talking to each other?

Answer. It doesn't matter. If there were multiple equilibria, talk might help the agents coordinate on a preferred equilibrium, but that is not the case here.

PROBLEMS FOR CHAPTER 9 Adverse Selection

9.2: Testing and Commitment. Fraction β of workers are talented, with output $a_t = 5$, and fraction $(1-\beta)$ are untalented, with output $a_u = 0$. Both types have a reservation wage of 1 and are risk neutral. At a cost of 2 to itself and 1 to the job applicant, employer Apex can test a job applicant and discover his true ability with probability θ, which takes a value of something over 0.5. There is just one period of work. Let $\beta = 0.001$. Suppose that Apex can commit itself to a wage schedule before the workers take the test, and that Apex must test all applicants and pay all the workers it hires the same wage, to avoid grumbling among workers and corruption in the personnel division.

(9.2a) What is the lowest wage, w_t, that will induce talented workers to apply? What is the lowest wage, w_u, that will induce untalented workers to apply? Which is greater?

Answer. If the firm cannot pay different wages depending on whether an applicant passes the test, it will not hire those who fail, since there would be no way to deter the untalented from applying. If the talented worker applies, he pays 1 for the test, his chance of getting the job and the wage w_t is θ, and his chance of not getting the job and settling for the reservation wage of 1 is $(1 - \theta)$, so his expected utility equals his reservation utility if $-1 + \theta w_t + (1 - \theta)(1) = 1$. Therefore, $w_t = 1 + \frac{1}{\theta}$. The expected utility of the untalented worker who applies equals his reservation utility if $-1 + \theta(1) + (1 - \theta)w_u = 1$, so $w_u = \frac{2-\theta}{1-\theta} = 1 + 1/(1 - \theta)$. Since $\theta > 0.5$, $w_t < w_u$. Untalented workers need a higher wage if they are to apply, because they fail the test more often.

The firm must pay at least $w = 1 + \frac{1}{\theta}$, the minimum wage acceptable to the talented, and if it pays that wage to those who pass the test, the

Answer. Without testing, the average quality of workers is 0.001(5) + 0.999(0), which is less than 1, so the firm will not hire at all if it does not test, and it will test if it can thereby make positive profits.
Apex will offer the wage of $w = 1 + \frac{1}{\theta}$, so only the talented workers will apply, and it will turn down any worker who fails the test. Apex's expected profit per worker who applies is then

$$\pi = -2 + \theta(5 - 1 - \tfrac{1}{\theta}.) \quad (20)$$

This exceeds zero if $\theta > 3/4$. Thus, $\underline{\theta} = 3/4$.

(9.2c) Now suppose that Apex can pay w_p to workers who pass the test and w_f to workers who flunk. What are w_p and w_f? What is the minimum accuracy value θ that will induce Apex to use the test? What are the firm's expected profits per worker who applies?

Answer. Apex would like to contrive a situation where only the talented apply and take the test, and where everyone who applies is hired at some wage or other. The self selection constraint that induces the talented to apply is

$$-1 + \theta w_p + (1-\theta) w_f \geq 1. \quad (21)$$

The self selection constraint that induces the untalented to not apply is

$$-1 + (1-\theta) w_p + \theta w_f \leq 1. \quad (22)$$

For the talented worker who has taken the test to be willing to take the job it must also be true that $w_p \geq 1$ and $w_f \geq 1$, unless the worker could somehow commit to taking the job once he had taken the test.

Apex will want to minimize the amount paid to the workers it hires, which means that inequality (21) will hold as an equality. There are many equally good wage combinations that do this while satisfying both constraints, all yielding the same profits. For example, let $w_f = 1$. Then from (21), $w_p = 1 + \frac{1}{\theta}$. Profit per worker who applies is

$$\pi \quad = -2 + 5 - \theta(1 - \tfrac{1}{\theta}) - (1-\theta)(1) = 3 \quad (23)$$

This only requires that $\theta > 0.5$, so the test need only be the slightest bit informative to be effective.

(9.2d) What happens if Apex cannot commit to paying the advertised wage, and can decide each applicant's wage individually?

Answer. If Apex cannot commit to wages ahead of time, then it will only pay 1 to a worker who has passed the test. Having already incurred the cost of testing, that cost is sunk for him, and he will accept the wage of 1, which yields a maximum utility of $-1 + 1 = 0$, and yields even less if he flunks. Foreseeing this trap, the worker would not apply.

Answer. If no untalented workers apply, Apex would deviate and save 2 by skipping the test and hiring every applicant, but then the untalented workers would start applying. If Apex tests every applicant, however, and pays only w_H, then no untalented worker will apply, and, again, Apex would deviate and skip the test.

9.4: Two-Time Losers. [10] Some people are strictly principled and will commit no robberies, even if there is no penalty. Others are incorrigible criminals and will commit two robberies, regardless of the penalty. Society wishes to inflict a certain penalty on criminals as retribution. Retribution requires an expected penalty of 15 per crime (15 if detection is sure, 150 if it has probability 0.1, etc.). Innocent people are sometimes falsely convicted, as shown in table 9.2.

Table 9.2: Two-Time Losers

	Convictions		
Robberies	0	1	2
0	0.81	0.18	0.01
2	0.49	0.42	0.09

Two systems are proposed: (i) a penalty of X for each conviction, and (ii) a penalty of 0 for the first conviction, and some amount P for the second conviction.

(9.4a) What must X and P be to achieve the desired amount of retribution?

Answer. The expected punishment of a criminal under system (i) is $0.42X + 0.09(2X) = 0.6X$. This must equal 30, so $X = 50$. The expected punishment of a criminal under system (ii) is $0.42(0)+0.09(P) = 0.09P$. This must equal 30, so $P = 333\frac{1}{3}$.

(9.4b) Which system inflicts the smaller cost on innocent people? How much is the cost in each case?

Answer. The expected cost under system (i) is 10 $(=0.18X+0.01(2X) = 0.2X)$. The expected cost under system (ii) is 3.33 $(=0.18(0)+0.01P)$. System (ii) has lower costs.

PROBLEMS FOR CHAPTER 9a Mechanism Design in Adverse Selection and in Moral Hazard with Hidden Information

9a.2: Task Assignment. Table 9a.1 shows the payoffs in the following but what she would like best is a job as Manager that gives her the freedom to choose rather than have the job designed for the task. The CEO of Rayco asks Sally which task is efficient. She can either reply "Task 1," "Task 2," or be silent. Her statement, if she makes one, is an example of "cheap talk," because it has no direct effect on anybody's payoff.[11]

Table 9a.1: The Right To Silence Game Payoffs

		Sally's Job		
		Job 1	Job 2	Manager
Sally knows:	Task 1 is efficient (.5)	2,5	$1,-2$	3,3
	Task 2 is efficient (.5)	$1,-2$	2,5	3,3

Payoffs to: (Sally, Rayco).

(9a.2a) If Sally did not have the option of speaking, what would happen?

Answer. Rayco would make her a Manager. Rayco's payoff is 3 then, but a deviation to either Job 1 or Job 2 would yield a payoff of .5(5) + .5(-2) = 1.5. Sally has no choices to make.

(9a.2b) There exist perfect Bayesian equilibria in which it does not matter how Sally replies. Find one of these in which Sally speaks at least some of the time, and explain why it is an equilibrium. You may assume that Sally is not morally or otherwise bound to speak the truth.

Answer. The key to answering this question and part (c) is to know what a perfect bayesian equilibrium is: a strategy for each player, plus any out-of-equilibrium beliefs that are needed. Someone who remembers that a strategy must specify what Sally does in each of the two states of the world and what Rayco does in response to each of Sally's three possible actions is a long ways towards answering the questions correctly. Here, try the following equilibrium:

Sally: Always say "Task 1." Rayco: Give Sally the job as Manager, regardless of her message. Out-of-equilibrium belief: Rayco thinks the probability that Task 1 is efficient is .5 if Sally says Task 2 or is silent.

Sally's payoff is 3, and she cannot change it by deviating. Rayco's payoff is 3, but a deviation to either Job 1 or Job 2 would yield a payoff of .5(5) + .5(-2) = 1.5.

This is an example of a "babbling equilibrium," so called because the uninformed player treats the informed player's cheap talk as meaning-

(9a.2c) There exists a perverse variety of equilibrium in which Sally always tells the truth and never is silent. Find an example of this equilibrium, and explain why neither player would have incentive to deviate to out-of-equilibrium behavior.

Answer. Sally: Say Task 1 if Task 1 is efficient. Say Task 2 if Task 2 is efficient. Rayco: If Sally says Task 1, give her Job 1. If Sally says Task 2, give her Job 2. If Sally is silent, give her Job 1. Out-of-equilibrium belief: If Sally is silent, then Task 1 is efficient.

Sally will tell the truth because if she deviates and the wrong task is assigned, her payoff will be 1 instead of 2. In particular, if she deviates and is silent, she will be given Job 1. Rayco has no incentive to deviate, because given that Sally always tells the truth, Rayco's payoff would fall from 5 to -2 from a deviation. If Sally is silent, which never happens in equilibrium, then Rayco's belief requires that Rayco give her Job 1 in order to maximize Rayco's payoff.

This out-of-equilibrium belief is not particularly plausible, and Farrell and Rabin use this as an example of an implausible equilibrium. It is good for learning how to describe equilibria, though!

9a.4: Incentive Compatibility and Price Discrimination. Two consumers have utility functions $u_1(x_1, y_1) = a_1 log(x_1) + y_1$ and $u_2(x_2, y_2) = a_2 log(x_2) + y_2$, where $1 > a_2 > a_1$. The price of the y-good is 1 and each consumer has an initial wealth of 15. A monopolist supplies the x-good. He has a constant marginal cost of 1.2 up to his capacity constraint of 10. He will offer at most two price-quantity packages, (r_1, x_1) and (r_2, x_2), where r_i is the total cost of purchasing x_i units. He cannot identify which consumer is which, but he can prevent resale.

(9a.4a) Write down the monopolist's profit maximization problem. You should have four constraints plus the capacity constraint.

Answer. The profit maximization problem is to maximize $r_1 + r_2 - 1.2(x_1 + x_2)$ subject to two participation constraints, two self selection constraints, and a capacity constraint:

$$\begin{aligned} a_1 log(x_1 - r_1 &\geq 0 \\ a_2 log(x_2 - r_2 &\geq 0 \\ a_1 log(x_1) - r_1 &\geq a_1 log(x_2) - r_2 \\ a_2 log(x_2) - r_2 &\geq a_2 log(x_1) - r_1 \\ 10 - x_1 - x_2 &\geq 0 \end{aligned}$$

(9a.4b) Which constraints will be binding at the optimal solution?

Answer. The binding constraints will be $a_1 log(x_1) - r_1 = 0$ and $a_2 log(x_2) -$

Consumer 2 values the good more, and hence will be made to pay the higher price for the larger number of units. The seller cannot take away all of Consumer 2's surplus, because Consumer 2 always has the option to pay the low price for the smaller number of units, but it can reduce Consumer 2's surplus to the point where he is indifferent between those alternatives. The seller can take away Consumer 1's surplus, however, so $a_1x_1 - r_1 = 0$.

(9a.4c) Substitute the binding constraints into the objective function. What is the resulting expression? What are the first-order conditions for profit maximization (you need not solve for the actual optimal values of the choice variables).

Answer. Profit comes to be $2a_1log(x_1) + a_2log(x_2) + a_2log(x_1)$.

PROBLEMS FOR CHAPTER 10 Signalling

10.2: Productive Education and Nonexistence of Equilibrium. Change Education I so that the two equally likely abilities are $a_L = 2$ and $a_H = 5$ and education is productive: the payoff of the employer whose contract is accepted is $\pi_{employer} = a + 2s - w$. The worker's utility function remains $U = w - \frac{8s}{a}$.

(10.2a) Under full information, what are the wages for educated and uneducated workers of each type, and who acquires education?

Answer. The wages paid will be equal to the worker's productivity: $w_{H0} = 5, w_{H1} = 7, w_{L0} = 2$, and $w_{L1} = 4$.

This being the case, only the high-ability workers will acquire education, because

$$U_H(y = 1) = 7 - 8/5 = 5.4 > U_H(y = 0) = 5 \qquad (24)$$

but

$$U_L(y = 1) = 4 - 8/2 = 0 < U_L(y = 0) = 2. \qquad (25)$$

(10.2b) Show that with incomplete information the equilibrium is unique (except for beliefs and wages out of equilibrium) but unreasonable.

Answer. The equilibrium is pooling, with zero education:

$$y_L = y_H = 0, w_0 = 3.5, w_1 = 4, Prob(L|(y = 1)) = 1, \qquad (26)$$

Profit is zero because $\pi_{employer} = 0.5(2 - 3.5) + 0.5(5 - 3.5))$. The worker's equilibrium payoffs are $\pi_H(y = 0) = \pi_L(y = 0) = 3.5$. $w_1 = 4$ because the employers believe that someone with education has low ability and a productivity of $2 + 2y$. This pooling equilibrium seems conjectures, $w_1 = 5.5(= 0.5(4) + 0.5(7))$, and when the wage is that high, the high-ability workers deviate to acquire education.

There cannot be a pooling equilibrium with $y = 1$, because then the average productivity would be 5.5 and the payoff to the Lows would be 1.5 $(= 5.5 - \frac{8}{2})$, less than the 2 they could get by deviating to $y = 0$. There cannot be a separating equilibrium, because the payoff to the Lows would be 2, which is less than the 3 $(= 7 - \frac{8}{2})$ they could get by deviating to $y = 1$. Thus, the peculiar pooling equilibrium is unique.

10.4: Signalling with a Continuous Signal. Suppose that with equal probability a worker's ability is $a_L = 1$ or $a_H = 5$, and the worker chooses any amount of education $y \in [0, \infty)$. Let $U_{worker} = w - \frac{8y}{a}$ and $\pi_{employer} = a - w$.

(10.4a) There is a continuum of pooling equilibria, with different levels of y^*, the amount of education necessary to obtain the high wage. What education levels, y^*, and wages, $w(y)$, are paid in the pooling equilibria, and what is a set of out-of-equilibrium beliefs that supports them? What are the incentive compatibility constraints?

<u>*Answer.*</u> A pooling equilibrium for any $y^* \in [0, 0.25]$ is

$$w = \left| \begin{array}{l} 1 \; if \; y \neq y^* \\ 3 \; if \; y = y^* \end{array} \right. \tag{27}$$

with the out-of-equilibrium belief that $Pr(L|(y \neq y^*)) = 1$, and with $y = y^*$ for both types.

The self selection constraints say that neither High nor Low workers want to deviate by acquiring other than y^* education. The most tempting deviation is to zero education, so the constraints are:

$$U_L(y^*) = w(y^*) - 8y^* \geq U_L(0) = w(y \neq y^*) \tag{28}$$

and

$$U_H(y^*) = w(y^*) - \frac{8y^*}{5} \geq U_H(0) = w(y \neq y^*). \tag{29}$$

The constraint on the Lows requires that $y^* \leq 0.25$ for a pooling equilibrium.

(10.4b) There is a continuum of separating equilibria, with different levels of y^*. What are the education levels and wages in the separating equilibria? Why are out-of-equilibrium beliefs needed, and what beliefs support the suggested equilibria? What are the self selection constraints for these equilibria?

<u>*Answer.*</u> A separating equilibrium for any $y^* \in [0.5, 2.5]$ is

$y = 0$ and $y = y^*$ occur in equilibrium, which leaves lots of other possibilities.

The self selection constraints say that High workers do not want to deviate by acquiring other than y^* of education (0 is most tempting), and the Lows do not want to deviate by acquiring y^* of education.

$$U_L(y^*) = w(y^*) - 8y^* \leq U_L(0) = w(y \neq y^*) \tag{31}$$

and

$$U_H(y^*) = w(y^*) - 8y^*/5 \geq U_H(0) = w(y \neq y^*). \tag{32}$$

These two constraints tell us that $y^* \geq 0.5$ and $y^* \leq 2.5$ in a separating equilibrium.

(10.4c) If you were forced to predict one equilibrium which will be the one played out, which would it be?

Answer. The out-of-equilibrium beliefs are unsatisfactory in the pooling equilibria because acquiring more than $y = 0.5$ in education is a dominated strategy for the Low type. If one carries this reasoning further, only $y = 0.5$ is satisfactory, because separating equilibria with more signalling require the belief that $y = 0.5$ is a sign of a Low type.

PROBLEMS FOR CHAPTER 11 Bargaining

11.2: Selling Cars. A car dealer must pay \$10,000 to the manufacturer for each car he adds to his inventory. He faces three buyers. From the point of view of the dealer, Smith's valuation is uniformly distributed between \$11,000 and \$21,000, Jones' is between \$9,000 and \$11,000, and Brown's is between \$4,000 and \$12,000. The dealer's policy is to make a single take-it-or-leave-it offer to each customer, and he is smart enough to avoid making differerent offers to customers who could resell to each other. Use the notation that the maximum valuation is $\overline{V}$ and the range of valuations is R.

(11.2a) What will the offers be?

Answer. Let us use units of thousands of dollars. The expected profit from a customer with maximum valuation $\overline{V} > 10$ and range of valuations R is, if price P is charged:

$$\pi(P; V, R) = \int_P^{\overline{V}} \frac{P-10}{R} dV$$

$$= \left(\frac{PV}{R} - \frac{10V}{R} \right) \Big|_P^{\overline{V}} \tag{33}$$

$$= \overline{V}P \quad 10\overline{V} \quad P^2 + 10P$$

so

$$P^* = \frac{\overline{V}}{2} + 5. \qquad (35)$$

Note that the optimal price does not depend on R, the range of possible valuations. Applying (35) to the specific customers: Smith will be offered $P = \frac{21}{2} + 5 = \$15,500$, Jones will be offered $P = \frac{11}{2} + 5 = \$10,500$, and Brown will be offered $P = \frac{12}{2} + 5 = \$11,000$. Moreover, Brown probably values the car less than Jones, but because of the higher probability that he values it more than \$10,000, he will end up paying more if he buys at all.

(11.2b) Who is most likely to buy a car? How does this compare with the outcome with perfect price discrimination under full information? How does it compare with the outcome when the dealer charges \$10,000 to each customer?

Answer. Smith will buy with probability 0.55, which is $\frac{21-15.5}{21-11}$. Jones will buy with probability 0.25. Brown will buy with probability 0.125. Thus, Smith is the buyer most likely to buy.

Whether the dealer charges \$10,000 or uses perfect price discrimination, the outcome is the same as far as allocative efficiency: Smith buys with probability 1, Jones buys with probability 0.5, and Brown buys with probability 0.25.

(11.2c) What happens to the equilibrium prices if, with probability 0.25, each buyer has a valuation of \$0, but the probability distribution remains otherwise the same?

Answer. The prices are the same as in part (a). If a buyer values the car at less than \$10,000, it is irrelevant what his value may be, since it is unprofitable to sell to him anyway. Only the part of his distribution above \$10,000 matters to the seller's strategy. Note that this has the same flavor as the analysis of auctions, where a bidder's strategy is conditioned on his having the highest valuation, since if he does not, he will generally lose the auction anyway and his bid is irrelevant.

11.4: Incomplete Information.

(11.1a) (11.4a) What is the equilibrium in the game of Bargaining with Incomplete Information if the probability of a low-valuation buyer is $\gamma = 0.1$, instead of 0.05 or 0.5?

Answer. In equilibrium, in the first period $p_1 = 150$, Buyer$_{100}$ accepts $p_1 \leq 100$, and Buyer$_{150}$ accepts p_1 with probability $m(p_1)$, where

$m = 1$ if $p_1 \leq 105$.

$m = \alpha$ if $105 < p_1 < 150$

sometimes accepted by Buyer_{150}, $p_2 = 150$ and is accepted by Buyer_{150}, and Buyer_{100} never accepts an offer.

The number $\frac{7}{9}$ is the m^* in equation (36) that equates the profit in the second period from reducing the price to 100 and maintaining it at 150.

$$\pi(p_2 = 100) = 100 = \pi(p_2 = 150) = \frac{0.9(1-m^*)}{0.1 + 0.9(1-m^*)}(150). \quad (36)$$

(11.4b) What level of γ marks the boundary between separating and pooling equilibria?

Answer. The boundary $\overline{\gamma}$ equals $\frac{1}{3}$. At that probability of a low-value buyer, even if it were known that zero high-valuation buyers had accepted the first-period offer, the seller would still not find it profitable to reduce his price to 100, because

$$\pi(p = 100) = 100 = \pi(p = 150) = \overline{\gamma}(0) + (1 - \overline{\gamma})(150). \quad (37)$$

11.6: A Fixed Bargaining Cost, Again. Apex and Brydox are entering into a joint venture that will yield 500 million dollars, but they must negotiate the split first. In bargaining round 1, Apex makes an offer at cost 0, proposing to keep A_1 for itself. Brydox either accepts (ending the game) or rejects. In Round 2, Brydox makes an offer at cost 10 million of A_2 for Apex, and Apex either accepts or rejects. In Round 3, Apex makes an offer of A_3 at cost c, and Brydox either accepts or rejects. If no offer is ever accepted, the joint venture is cancelled.

(11.6a) If $c = 0$, what is the equilibrium? What is the equilibrium outcome?

Answer. Apex: Offer $A_1 = 500$, accept $A_2 \geq 500$, and offer $A_3 = 500$. Brydox: Accept $A_1 \leq 500$, offer $A_2 = 500$, and accept $A_3 \leq 500$. Outcome: $A_1 = 500$, and it is accepted.

(11.6b) If $c = 10$, what is the equilibrium? What is the equilibrium outcome?

Answer. Apex: Offer $A_1 = 500$, accept $A_2 \geq 490$, and offer $A_3 = 500$. Brydox: Accept $A_1 \leq 500$, offer $A_2 = 490$, and accept $A_3 \leq 500$. Outcome: $A_1 = 500$, and it is accepted.

(11.6c) If $c = 300$, what is the equilibrium? What is the equilibrium outcome?

Answer. Apex: Offer $A_1 = 210$, accept $A_2 \geq 200$, and offer $A_3 = 500$. Brydox: Accept any $A_1 \leq 210$, offer $A_2 = 200$, and accept $A_3 \leq 500$. Outcome: $A_1 = 210$, and it is accepted.

12.2: The Founding of Hong Kong.[12] The Tai-Pan and Mr. Brock are bidding in an English auction for a parcel of land on a knoll in Hong Kong. They must bid integer values, and the Tai-Pan bids first. Tying bids cannot be made, and bids cannot be withdrawn once they are made. The direct value of the land is 1 to Brock and 2 to the Tai-Pan, but the Tai-Pan has said publicly that he wants it, so if Brock gets it, he receives 5 in "face" and the Tai-Pan loses 10. Moreover, Brock hates the Tai-Pan and receives 1 in utility for each 1 that the Tai-Pan pays out to get the land.

(12.2a) Fill in the entries in Table 12.2.

Table 12.2 The Tai-Pan Game

Winning bid:	1	2	3	4	5	6	7	8	9	10	11	12
If Brock wins:												
π_{Brock}												
$\pi_{Tai-Pan}$												
If Brock loses:												
π_{Brock}												
$\pi_{Tai-Pan}$												

Answer. See Table C.9.

Table C.9 "The Tai-Pan Game"

Winning bid:	1	2	3	4	5	6	7	8	9	10	11	12
If Brock wins:												
π_{Brock}	5	4	3	2	1	0	-1	-2	-3	-4	-5	-6
$\pi_{Tai-Pan}$	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
If Brock loses:												
π_{Brock}	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{Tai-Pan}$	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10

(12.2b) In equilibrium, who wins, and at what bid?

Answer. There are two possible equilibrium outcomes: (i) The Tai-Pan wins with a bid of 10. The Tai-Pan bids 10 immediately, and is willing to bid up to 11. Brock is willing to bid up to 10. If Brock were to bid 11 and win, the Tai-Pan's payoff would be -10, which is no higher than if the Tai-Pan overbid to win at 12. (ii) The Tai-Pan wins with a bid of 11. The Tai-Pan bids 11 immediately, and is willing to bid up to 12. Brock is willing to bid up to 11. The outcome is that the Tai-Pan wins at 11, for a payoff of -9 (=-11 + 2). If the Tai-Pan were to deviate by bidding just 10, then Brock would bid 11 and the Tai-Pan would overbid to win at 12, for a payoff of $-10 (= -12 + 2)$.

The essence of this problem is that winning is not the only reason to win with a bid of 3, because if Brock bids any higher, Brock will win and have a lower payoff than if he loses to the Tai-Pan's bid of 3." The reason this is false is that Brock does not intend to win when he bids higher than 3: he intends to lose anyway, but to harm the Tai-Pan more.

(12.2c) What happens if the Tai-Pan can precommit to a strategy?

Answer. The Tai-Pan can then precommit not to let Brock engage him in a malicious bidding war. The Tai-pan commits to pay no more than 3, and bids that immediately. Brock will not bid up to 4, because he would win, with a payoff of 2 from winning at 4 compared to 3 from losing at 3.

(12.2d) What happens if the Tai-Pan cannot precommit, but he also hates Brock, and gets 1 in utility for each 1 that Brock pays out to get the land?

Answer. The Tai-pan will bid up to 6. Brock bids up to 5. Thus, the Tai-Pan will bid 5 immediately, and win at that price.[13]

Problem 12.4: An Auction with Stupid Bidders. Smith's value for an object has a private component equal to 1 and component common with Jones and Brown. Jones's and Brown's private components both equal zero. Each player estimates the common component Z independently, and player i's estimate is either x_i above the true value or x_i below, with equal probability. Jones and Brown are naive and always bid their valuations. The auction is English.

(12.4a) If $x_{Smith} = 0$, what is Smith's dominant strategy if his estimate of Z equals 20?

Answer. Bid up to 21.

(12.4b) If $x_i = 8$ for all players and Smith estimates $Z = 20$, what are the probabilities that he puts on different values of Z?

Answer. $Prob(Z = 12) = Prob(Z = 28) = 0.5$. $Prob(Z \notin \{12, 28\}) = 0$.

(12.4c) If $x_i = 8$ but Smith knows that $Z = 12$ with certainty, what are the probabilities he puts on the different combinations of bids by Jones and Brown?

Answer. $Prob(4, 4) = Prob(4, 20) = Prob(20, 4) = Prob(20, 20) = 0.25$.

12.4d) (Why is 9 a better upper limit on bids for Smith than 21, if his estimate

Answer. First, consider bidding up to 9. With probability 0.5, the true value is just $Z = 12$, and in that case with probability 0.25 both Jones and Brown underestimate the value, and estimate it to equal 4. This is the only circumstance under which Smith could win with a bid as low as 9. He will win the auction at a price of 4, and earn a payoff of $1.125 (= 0.5 \cdot 0.25(13-4))$. Next, consider bidding up to 21. Again, with probability $0.5 \cdot 0.25$ the true value is $V = 12$ and Smith wins at a price of 4. But now with probability $0.5 \cdot 0.75$ the true value is $V = 12$ and Smith wins at a price of 20. Also, even if the true value is $V = 28$, which has probability 0.5, Smith wins if both Jones and Brown underestimate the value. Thus, Smith's payoff from the strategy of bidding up to 21 is $-0.375(= 0.5 \cdot 0.25(13-4) + 0.5 \cdot 0.75(13-20) + 0.5 \cdot 0.25(29-20))$.

PROBLEMS FOR CHAPTER 13 Pricing

13.2: Cournot Mergers. (See Salant, Switzer, & Reynolds [1983].) There are three identical firms in an industry with demand given by $P = 1 - Q$, where $Q = q_1 + q_2 + q_3$. The marginal cost is zero.

(13.2a) Compute the Cournot equilibrium price and quantities.

Answer.

$$\pi_1 = (1 - q_1 - q_2 - q_3)q_1 \tag{38}$$

Maximizing this gives $(1-q_2-q_3)-2q_1 = 0$. In a symmetric equilibrium, $q_1 = q_2 = q_3$, so $1 - 2q - 2q = 0$, and $q = 0.25$. The price is then $1 - 3q = 0.25$.

((13.2b) How do you know that there are no asymmetric Cournot equilibria, in which one firm produces a different amount than the others?

Answer. Each firm has a linear reaction curve, so the reaction curves cannot all intersect at more than one point unless the reaction curves are identical, which they certainly are not. We have shown that there exists one equilibrium which is symmetric, so there cannot be any other equilibrium.

Another way to see this is by using algebra. The first-order conditions are

$$\begin{aligned} 2q_1 + q_2 + q_3 &= 1 \\ q_1 + 2q_2 + q_3 &= 1 \\ q_1 + q_2 + 2q_3 &= 1 \end{aligned} \tag{39}$$

Subtracting the second of these equations from the first gives $q_1 - q_2 = 0$. Subtracting the third from the second gives $q_2 - q_3 = 0$. Hence, $q_1 = q_2 = q_3$ in equilibrium unless there are corner solutions, of which none exist here.

If the number of firms falls to two, we must solve the new Cournot game. Now, the first order condition is $(1 - q_2) - 2q_1 = 0$, so in a symmetric equilibrium $q = \frac{1}{3}$. The price is then $1 - 2q = \frac{1}{3}$. Profit per firm is about $0.111 (= (\frac{1}{3})(\frac{1}{3}) = \frac{1}{9})$. Thus, the merged firm has lower profits than its two component firms used to have.

Problem 13.4: Asymmetric Cournot Duopoly. Apex has variable costs of q_a^2 and a fixed cost of 1000, while Brydox has variables costs of $2q_b^2$ and no fixed cost. Demand is $p = 115 - q_a - q_b$.

(13.4a) What is the equation for Apex's Cournot reaction function?

Answer. This is found from the first order condition for Apex's maximization problem,

$$\underset{q_a}{Maximize} \;\; (115 - q_a - q_b)q_a - 1000 - q_a^2, \tag{40}$$

which yields $q_a = 28.75 - 0.25q_b$.

(13.4b) What is the equation for Brydox' Cournot reaction function?

Answer. This is found from the first order condition for Brydox's maximization problem,

$$\underset{q_b}{Maximize} \;\; (115 - q_a - q_b)q_b - 2q_b^2, \tag{41}$$

which yields $q_b = \frac{115}{6} - \frac{q_a}{6}$.

(13.4c) What are the outputs and profits in the Cournot equilibrium?

Answer. Solving the reaction functions together yields $q_a = 25$ and $q_b = 15$. The demand and cost curves can then be used to find that $\pi_a = 250$ and $\pi_b = 675$.

PROBLEMS FOR CHAPTER 14 Entry

14.2: Rent Seeking. I mentioned that Rogerson (1982) uses a game very similar to "Patent Race for a New Market" to analyze competition for a government monopoly franchise. See if you can do this too. What can you predict about the welfare results of such competition?

_Answer._xxx

14.4: Entry for Buyout. Find the equilibrium in Entry for Buyout if all the parameters of the numerical example are the same except that the