September 11, 1999.

9 Adverse Selection

9.1 Introduction: Production Game VI

"Production Game VI : Adverse Selection"

Players
The principal and the agent.

The Order of Play

(0) Nature chooses the agent's ability a, unobserved by the principal, according to distribution $F(a)$.

(1) The principal offers the agent one or more wage contracts $w_1(q), w_2(q), \ldots$

(2) The agent accepts one contract or rejects them all.

(3) Nature chooses a value for the state of the world, θ, according to distribution $G(\theta)$. Output is then $q = q(a, \theta)$.

Payoffs
If the agent rejects all contracts, then $\pi_{agent} = \overline{U}$ and $\pi_{principal} = 0$.
Otherwise, $\pi_{agent} = U(w)$ and $\pi_{principal} = V(q - w)$.

"Production Game VIa: Adverse Selection, with Particular Parameters"

Players
The principal and the agent.

The Order of Play

(0) Nature chooses the agent's ability a, unobserved by the principal, according to distribution $F(a)$, which puts probability 0.9 on low ability $a = 0$ and probability 0.1 on high ability $a = 10$.

(1) The principal offers the agent one or more wage contracts $\{w_1(q = 0), w_1(q = 10)\}, \{w_2(q = 0), w_2(q = 10)\} \ldots$

(2) The agent accepts one contract or rejects them all.

(3) Nature chooses a value for the state of the world, θ, according to distribution $G(\theta)$, which puts equal weight on 0 and 10. Output is then $q = Max(a + \theta, 10)$.

Payoffs
If the agent rejects all contracts, then $\pi_{agent} = 4$ and $\pi_{principal} = 0$.
Otherwise, $\pi_{agent} = w$ and $\pi_{principal} = q - w$.

An equilibrium is

$$
\begin{array}{l}
Principal: Offer\ \{w_1(q=0)=0, w_1(q=10)=4\}, \{w_2(q=0)=4, w_2(q=10)\\
Low\ agent:\ \ Accept\ w_2.\\
High\ agent:\ \ Accept\ w_1.
\end{array}
$$

The principal's problem, as in Production Game V, is to maximize his profits subject to

(1) **Incentive compatibility** (the agent picks the desired contract and actions).

(2) **Participation** (the agent prefers the contract to his reservation utility).

In a model with hidden knowledge, the incentive compatibility constraint is customarily called the **self-selection constraint**, because it induces the different types of agents to pick different contracts. Here, however, there will be an entire set of constraints, one for each type of agent, since all the types have different incentives. In Production Game VI, the self selection constraints are

$$
\begin{array}{l}
U_L(W_2) \geq U_L(W_1);\ \ 0.5(4)+0.5(4) \geq 0.5(0)+0.5(4) \\
U_H(W_1) \geq U_H(W_2);\ so\ 0.5(4)+0.5(4) \geq 0.5(0)+0.5(4)
\end{array} \tag{1}
$$

The risky wage contract W1 has to have a low enough expected return for the Low agents to deter them from accepting it; but the safe wage contract W2 must be less attractive than W1 to the High agents.

The participation constraints are

$$
\begin{array}{l}
U_L(W_2) \geq \overline{U_L};\ \ 0.5(4)+0.5(4) \geq 4 \\
U_H(W_1) \geq \overline{U_H};\ so\ 0.5(4)+0.5(4) \geq 4
\end{array} \tag{2}
$$

If all types of agents choose the same strategy in all states, the equilibrium is **pooling**. *Otherwise, it is* **separating**.

The distinction between pooling and separating is different from the distinction between equilibrium concepts.

These two terms came up in Section 6.2 in the game of PhD Admissions.

A separating contract need not be fully separating. If agents who observe $\theta \leq 4$ accept contract C_1 but other agents accept C_2, then the equilibrium is separating but it does not separate out every type. We say that the equilibrium is **fully revealing** if the agent's choice of contract always conveys his private information to the principal. Between pooling and fully revealing equilibria are the **imperfectly separating** equilibria synonymously called **semi-separating**, **partially separating**, **partially revealing**, or **partially pooling** equilibria.

9.2 Adverse Selection under Certainty: Lemons I and II

"The Basic Lemons Model"

Players
A buyer and a seller.

The Order of Play

(0) Nature chooses quality type θ for the seller according to the distribution $F(\theta)$.
The seller knows θ, but while the buyer knows F, he does not know the θ of the particular seller he faces.

(1) The buyer offers a price P.

(2) The seller accepts or rejects.

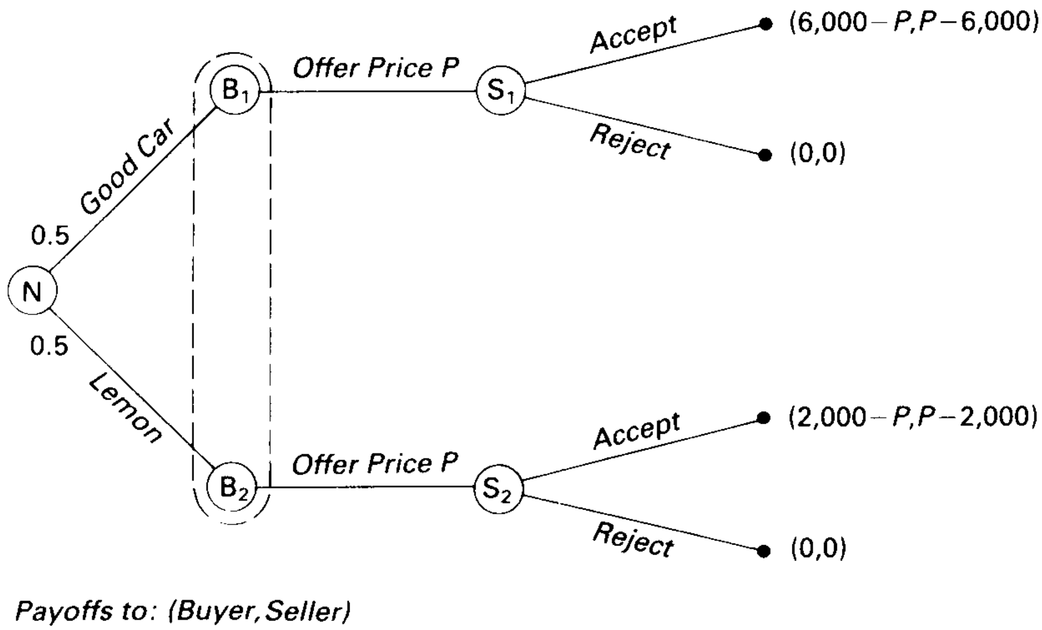
Payoffs
If the buyer rejects the offer, both players receive payoffs of zero. Otherwise, $\pi_{buyer} = V(\theta) - P$ and $\pi_{seller} = P - U(\theta)$, where V and U will be defined later.

Lemons I: Identical Tastes, Two Types of Sellers

Let good cars have quality $6,000$ and bad cars (lemons) quality $2,000$, so $\theta \in \{2,000,\ 6,000\}$, and suppose that half the cars in the world are of the first type and the other half of the second type. A payoff profile of (0,0) will represent the status quo, in which the buyer has $50,000 and the seller has the car. Assume that both players are risk neutral and they value quality at one dollar per unit, so after a trade the payoffs are $\pi_{buyer} = \theta - P$ and $\pi_{seller} = P - \theta$. The extensive form is shown in Figure 9.1.

Figure 9.1: An Extensive Form for Lemons I

Figure 8.1 Lemons I



Lemons II: Identical Tastes, a Continuum of Types of Sellers

We will assume that the quality types are uniformly distributed between 2,000 and 6,000. The average quality is $\overline{\theta} = 4,000$, which is therefore the price the buyer would be willing to pay for a car of unknown quality if all cars were on the market. The probability density is zero except on the support [2,000, 6,000], where it is $f(\theta) = 1/(6,000-2,000)$, and the cumulative density is

$$F(\theta) = \int_{2,000}^{\theta} f(x)dx. \tag{3}$$

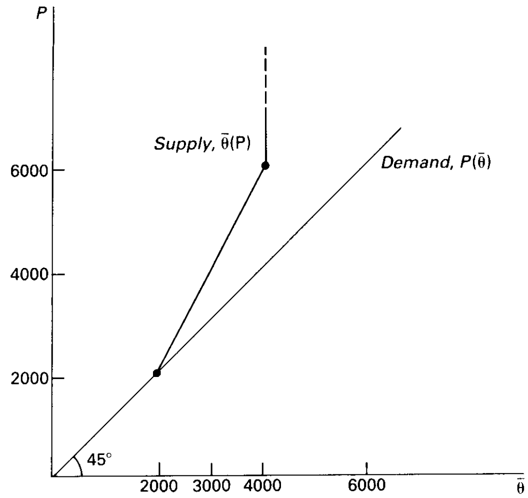
After substituting the uniform density for $f(\theta)$ and integrating (1) we obtain

$$F(\theta) = \frac{\theta}{4,000} - 0.5. \tag{4}$$

The payoff functions are the same as in Lemons I.

Figure 9.2: Lemons II: Identical Tastes

Figure 8.2 Lemons II: identical tastes



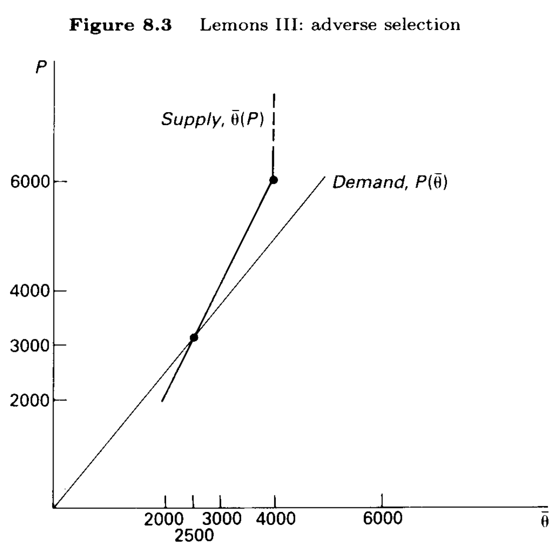
9.3 Heterogeneous Tastes: Lemons III and IV

The outcome that no cars are traded is extreme, but there is no efficiency loss in either Lemons I or Lemons II. Since all the players have identical tastes, it does not matter who ends up owning the cars.

Lemons III : Buyers Value Cars More than Sellers

Assume that sellers value their cars at exactly their qualities θ, but that buyers have valuations 20 percent greater, and, moreover, outnumber the sellers. The payoffs if a trade occurs are $\pi_{buyer} = 1.2\theta - P$ and $\pi_{seller} = P - \theta$. In equilibrium, the sellers will capture the gains from trade.

Figure 9.3: Adverse Selection When Buyers Value Cars More Than Sellers: Lemons III



bigskip **Lemons IV : Sellers' Valuations Differ**

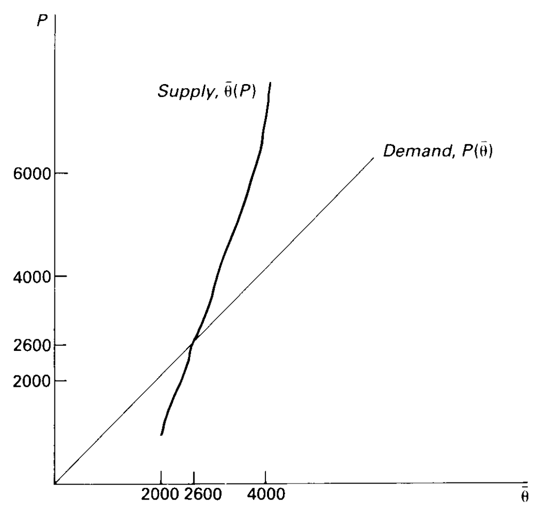
For a particular seller, the valuation of one unit of quality is $1 + \varepsilon$, where the random disturbance ε can be either positive or negative and has an expected value of zero. The disturbance could arise because of the seller's mistake— he did not realize how much he would enjoy driving when he bought the car— or because conditions changed— he switched to a job closer to home. Payoffs if a trade occurs are $\pi_{buyer} = \theta - P$ and $\pi_{seller} = P - (1+\varepsilon)\theta$.

If $\varepsilon = -0.15$ and $\theta = 2,000$, then \$1,700 is the lowest price at which the player would resell his car. The average quality of cars offered for sale at price P is the expected quality of cars valued by their owners at less than P, i.e.,

$$\overline{\theta}(P) = E\left(\theta \mid (1+\varepsilon)\theta \leq P\right). \tag{5}$$

Figure 9.4: Lemons IV: Sellers' Valuations Differ

Figure 8.4 Lemons IV: sellers' valuations differ



The equilibrium drawn in Figure 9.4 is ($P = \$2,600, \overline{\theta} = 2,600$).

Heterogeneous Buyers: Excess Supply

If buyers have different valuations for quality, the market might not clear, as C. Wilson (1980) points out. Assume that the number of buyers willing to pay \$1.2 per unit of quality exceeds the number of sellers, but that buyer Smith is an eccentric whose demand for high quality is unusually strong. He would pay \$100,000 for a car of quality 5,000 or greater, and \$0 for a car of any lower quality.

In Lemons III without Smith, the outcome is a price of \$3,000, an average market quality of 2,500, and a market quality range between 2,000 and 3,000. Smith would be unhappy with this, since he has zero probability of finding a car he likes. In fact, he would be willing to accept a price of \$6,000, so that all the cars, from quality 2,000 to 6,000, would be offered for sale and the probability that he buys a satisfactory car would rise from 0 to 0.25. But Smith would not want to buy all the cars offered to him, so the equilibrium has two prices, \$3,000 and \$6,000, with excess supply at the higher price.

Strangely enough, Smith's demand function is upward sloping. At a price of \$3,000, he is unwilling to buy; at a price of \$6,000, he is willing, because expected quality rises with price. This does not contradict basic price theory, for the standard assumption of *ceteris paribus* is violated. As the price increases, the quantity demanded would fall if all else stayed the same, but all else does not— quality rises.

9.4 Adverse Selection under Uncertainty: Insurance Game III

"Insurance Game III"

Players
Smith and two insurance companies.

The Order of Play

(0) Nature chooses Smith to be either $Safe$, with probability 0.6, or $Unsafe$, with probability 0.4. Smith knows his type, but the insurance companies do not.

(1) Each insurance company offers its own contract (x, y) under which Smith pays premium x unconditionally and receives compensation y if there is a theft.

(2) Smith picks a contract.

(3) Nature chooses whether there is a theft, using probability 0.5 if Smith is $Safe$ and 0.75 if he is $Unsafe$.

Payoffs.
Smith's payoff depends on his type and the contract (x, y) that he accepts. Let $U' > 0$ and $U'' < 0$.

$$\pi_{Smith}(Safe) = 0.5U(12 - x) + 0.5U(0 + y - x).$$

$$\pi_{Smith}(Unsafe) = 0.25U(12 - x) + 0.75(0 + y - x).$$

The companies' payoffs depend on what types of customers accept their contracts, as shown in table 9.1.

Table 9.1 Insurance Game III: Payoffs

Company payoff	Types of customers
0	no customers
$0.5x + 0.5(x-y)$	just *Safe*
$0.25x + 0.75(x-y)$	just *Unsafe*
$0.6[0.5x + 0.5(x-y)] + 0.4[0.25x + 0.75(x-y)]$	*Unsafe* and *Safe*

Smith is *Safe* with probability 0.6 and *Unsafe* with probability 0.4. Without insurance, Smith's dollar wealth is 12 if there is no theft and 0 if there is, depicted in Figure 9.5 as his endowment in state space, $\omega = (12, 0)$. If Smith is *Safe*, a theft occurs with probability 0.5, but if he is *Unsafe* the probability is 0.75. Smith is risk averse (because $U'' < 0$) and the insurance companies are risk neutral.

Figure 9.5: Insurance Game III: Nonexistence of a Pooling Equilibrium

Figure 8.5 Insurance Game III: nonexistence of pooling equilibrium

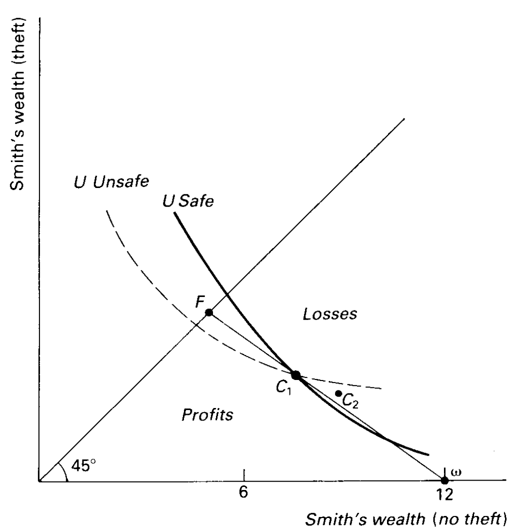


Figure 9.6: A Separating Equilibrium for Insurance Game III

Figure 8.6 Insurance Game III: a separating equilibrium

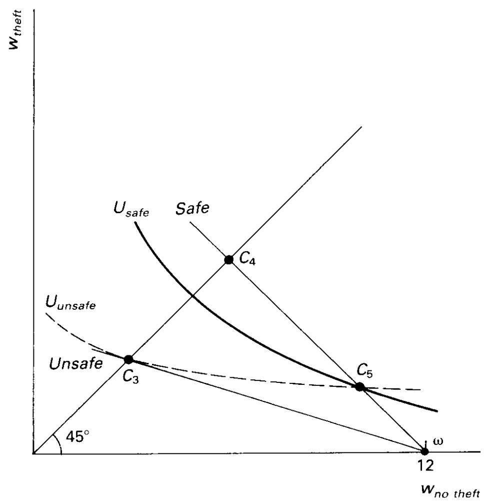
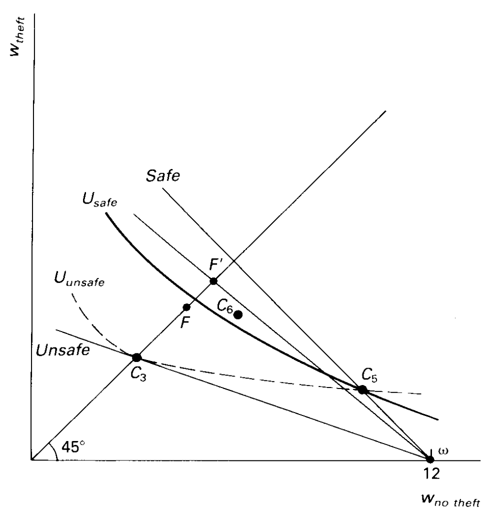


Figure 9.7: Curves for Which There is No Equilibrium in Insurance Game III

Figure 8.7 Insurance Game III: no equilibrium exists



***9.5 Market Microstructure** (formerly Section 15.3)

"The Bagehot Model"

Players
The informed trader and two competing marketmakers.

The Order of Play

(0) Nature chooses the asset value v to be either $p_0 - \delta$ or $p_0 + \delta$ with equal probability. The marketmakers never observe the asset value, nor do they observe whether anyone else observes it, but the "informed" trader observes v with probability θ.

(1) The marketmakers choose their spreads s, offering prices $p_{bid} = p_0 - \frac{s}{2}$ at which they will buy the security and $p_{ask} = p_0 + \frac{s}{2}$ for which they will sell it.

(2) The informed trader decides whether to buy one unit, sell one unit, or do nothing.

(3) Noise traders buy n units and sell n units.

Payoffs
Everyone is risk neutral. The informed trader's payoff is $v - p_{ask}$ if he buys, $p_{bid} - v$ if he sells, and zero if he does nothing. The marketmaker who offers the highest p_{bid} trades with all the customers who wish to sell, and the marketmaker who offers the lowest p_{ask} trades with all the customers who wish to buy. If the marketmakers set equal prices, they split the market evenly. A marketmaker who sells x units gets a payoff of $x(p_{ask} - v)$, and a marketmaker who buys x units gets a payoff of $x(v - p_{bid})$.

This is a very simple game. Competition between the marketmakers will make their prices identical and their profits zero. The informed trader should buy if $v > p_{ask}$ and sell if $v < p_{bid}$. He has no incentive to trade if $[p_{bid}, p_{ask}]$.

A marketmaker will always lose money trading with the informed trader, but if $s > 0$, so $p_{ask} > p_0$ and $p_{bid} < p_0$, he will earn positive expected profits trading with the noise traders. Since a marketmaker could specialize in either sales or purchases, he must earn zero expected profits overall from either type of trade. Centering the bid-ask spread on the expected value of the stock, p_0, ensures this. Marketmaker sales will be at the ask price of $(p_0 + s/2)$. With probability 0.5, this is above the true value of the stock, $(p_0 - \delta)$, in which case the informed trader will not buy but the marketmakers will earn a total profit of $n[(p_0 + s/2) - (p_0 - \delta)]$ from the noise traders. With probability 0.5, the ask price of $(p_0 + s/2)$ is below the true value of the stock, $(p_0 + \delta)$, in which case the informed trader will be informed with probability θ and buy one unit and the noise traders will buy n more in any case, so the marketmakers will earn a total expected profit of $(n + \theta)[(p_0 + s/2) - (p_0 + \delta)]$, a negative number. For marketmaker profits from sales at the ask price to be zero overall, this expected profit must be set to zero:

$$.5n[(p_0 + s/2) - (p_0 - \delta)] + .5(n + \theta)[(p_0 + s/2) - (p_0 + \delta)] = 0 \quad (6)$$

This equation implies that $n[s/2 + \delta] + (n + \theta)[s/2 - \delta] = 0$, so

$$s^* = \frac{2\delta\theta}{2n + \theta}. \quad (7)$$

The profit from marketmaker purchases must similarly equal zero, and will for the same spread s^*, though we will not go through the algebra here. The zero profit equation is

$$.5(n+\theta)[(p_0 - \delta) - (p_0 - s/2) -] + .5n[(p_0 + \delta) - (p_0 - s/2)] = 0. \quad (8)$$

Equation (9.7) has a number of implications. First, the spread s^* is positive. Even though marketmakers compete and have zero transactions costs, they charge a different price to buy and to sell. They make money dealing with the noise traders but lose money with the informed trader, if he is present. The comparative statics reflect this. s^* rises in δ, the variance of the true value, because divergent true values increase losses from trading with the informed trader, and s^* falls in n, which reflects the number of noise traders relative to informed traders, because when there are more noise traders, the profits from trading with them are greater. s^* rises in θ, the probability that the informed trader really has inside information, which is also intuitive but requires a little calculus to demonstrate starting from equation (9.7):

$$\frac{\partial s^*}{\partial \theta} = \frac{2\delta}{2n+\theta} - \frac{2\delta\theta}{(2n+\theta)^2} = \left(\frac{1}{(2n+\theta)^2}\right)(4\delta n + 2\delta\theta - 2\delta\theta) > 0. \tag{9}$$

"The Kyle Model" (Kyle [1985])

Players
The informed trader and two competing marketmakers.

The Order of Play

(0) Nature chooses the asset value v from a normal distribution with mean p_0 and variance σ_v^2, observed by the informed trader but not by the marketmakers.

(1) The informed trader offers a trade of size $x(v)$, which is a purchase if positive and a sale if negative, unobserved by the marketmaker.

(2) Nature chooses a trade of size u by noise traders, unobserved by the marketmaker, where u is distributed normally with mean zero and variance σ_u^2.

(3) The marketmakers observe the total market trade offer $y = x + u$, and choose prices $p(y)$.

(4) Trades are executed. If y is positive (the market wants to purchase, in net), whichever marketmaker offers the lowest price executes the trades; if y is negative (the market wants to sell, in net), whichever marketmaker offers the highest price executes the trades. v is then revealed to everyone.

Payoffs
All players are risk neutral. The informed trader's payoff is $(v-p)x$. The marketmaker's payoff is zero if he does not trade and $(p-v)y$ if he does.

An equilibrium for this game is the strategy profile

$$x(v) = (v - p_0)\left(\frac{\sigma_u}{\sigma_v}\right) \tag{10}$$

and

$$p(y) = p_0 + \left(\frac{\sigma_v}{2\sigma_u}\right) y. \tag{11}$$

I will not (and cannot) prove uniqueness of the equilibrium, since it is very hard to check all possible profiles of nonlinear strategies, but I will show that $\{(9.10),(9.11)\}$ is Nash and is the unique linear equilibrium. To start, hypothesize that the informed trader uses a linear strategy, so

$$x(v) = \alpha + \beta v \tag{12}$$

for some constants α and β. Competition between the market-makers means that their expected profits will be zero, which requires that the price they offer be the expected value of v. Thus, their equilibrium strategy $p(y)$ will be an unbiased estimate of v given their data y, where they know that y is normally distributed and that

$$\begin{aligned} y &= x + u \\ &= \alpha + \beta v + u. \end{aligned} \tag{13}$$

This means that their best estimate of v given the data y is, following the usual regression rule (which readers unfamiliar with statistics must accept on faith),

$$\begin{aligned} E(v|y) &= E(v) + \left(\frac{cov(v,y)}{var(y)}\right) y \\ &= p_0 + \left(\frac{\beta\sigma_v^2}{\beta^2\sigma_v^2+\sigma_u^2}\right) y \\ &= p_0 + \lambda y, \end{aligned} \tag{14}$$

where λ is a new shorthand variable to save writing out the term in parentheses in what follows.

The function $p(y)$ will be a linear function of y under our assumption that x is a linear function of v. Given that $p(y) = p_0+\lambda y$, what must next be shown is that x will indeed be a linear function of v. Start by writing the informed trader's expected payoff, which is

$$
\begin{aligned}
E\pi_i &= E([v - p(y)]x) \\
&= E([v - p_0 - \lambda(x + u)]x) \qquad (15) \\
&= [v - p_0 - \lambda(x + 0)]x,
\end{aligned}
$$

since $E(u) = 0$. Maximizing the expected payoff with respect to x gives the first order condition

$$v - p_0 - 2\lambda x = 0, \qquad (16)$$

which on rearranging becomes

$$x = -\frac{p_0}{2\lambda} + \left(\frac{1}{2\lambda}\right) v. \qquad (17)$$

Equation (9.17) establishes that $x(v)$ is linear, given that $p(y)$ is linear. All that is left is to find the value of λ. See by comparing (9.17) and (9.12) that $\beta = \frac{1}{2\lambda}$. Substituting this β into the value of λ from (9.14) gives

$$\lambda = \frac{\beta\sigma_v^2}{\beta^2\sigma_v^2 + \sigma_u^2} = \frac{\frac{\sigma_v^2}{2\lambda}}{\frac{\sigma_v^2}{(4\lambda^2)} + \sigma_u^2}, \qquad (18)$$

which upon solving for λ yields $\lambda = \frac{\sigma_v}{2\sigma_u}$. Since $\beta = \frac{1}{2\lambda}$, it follows that $\beta = \frac{\sigma_u}{\sigma_v}$. These values of λ and β together with equation (9.17) give the strategies asserted at the start in equations (9.10) and (9.11).