13 Pricing

September 6, 1999 chapter. October 16, 1999 overheads. Eric Rasmusen, Erasmuse@indiana.edu. Web: Php.indiana.edu/~erasmuse.

13.1 Quantities as Strategies: Cournot Equilibrium Revisited

Apex sees the effect of its output on price as

$$\frac{\partial p}{\partial q_a} = \frac{dp}{dq}\frac{\partial q}{\partial q_a} = \frac{dp}{dq}. \tag{1}$$

Apex's payoff function is

$$\pi_a = p(q)q_a - c(q_a). \tag{2}$$

To find Apex's reaction function, we differentiate with respect to its strategy to obtain

$$\frac{d\pi_a}{dq_a} = p + \frac{dp}{dq}q_a - \frac{dc}{dq_a} = 0, \tag{3}$$

which implies

$$q_a = \frac{\frac{dc}{dq_a} - p}{\frac{dp}{dq}}, \tag{4}$$

or, simplifying the notation,

$$q_a = \frac{c' - p}{p'}. \tag{5}$$

To find the change in Apex's best response for an exogenous change in Brydox's output, differentiate (13.5) with respect to q_b, remembering that q_b exerts not only a direct effect, but possibly an indirect effect on q_a.

$$\frac{dq_a}{dq_b} = \frac{(p-c')(p'' + p''\frac{dq_a}{dq_b})}{p'^2} + \frac{c''\frac{dq_a}{dq_b} - p' - p'\frac{dq_a}{dq_b}}{p'}. \tag{6}$$

Equation (13.6) can be solved for $\frac{dq_a}{dq_b}$ to obtain the slope of the reaction function,

$$\frac{dq_a}{dq_b} = \frac{(p-c')p'' - p'^2}{2p'^2 - c''p' - (p-c')p''} \tag{7}$$

If both costs and demand are linear, as in Section 3.5, then $c'' = 0$ and $p'' = 0$, so equation (13.7) becomes

$$\frac{dq_a}{dq_b} = -\frac{p'^2}{2p'^2} = -\frac{1}{2}. \tag{8}$$

Nonexistence and Multiplicity of Equilibria

$$\frac{dq_a}{dq_b} = \frac{(p - c')p'' - p'^2}{2p'^2 - c''p' - (p - c')p''}$$

The general model faces two problems that did not arise in the linear model: nonuniqueness and nonexistence. If demand is concave and costs are convex, which implies that $p'' < 0$ and $c'' > 0$, then all is well as far as existence goes. Since price is greater than marginal cost ($p > c'$), equation (13.7) tells us that the reaction functions are downward sloping, because $2p'^2 - c''p' - (p - c')p''$ is positive and both $(p - c')p''$ and $-p'^2$ are negative. If the reaction curves are downward sloping, they cross and an equilibrium exists, as was shown in Figure 3.1 for the linear case represented by equation (13.8).

If demand is convex or costs are concave, so $p'' > 0$ or $c'' < 0$, the reaction functions can be upward sloping, in which case they might never cross and no equilibrium would exist. The problem can also be seen from Apex' payoff function, equation (13.2). If $p(q)$ is convex, the payoff function might not be concave, in which case standard maximization techniques break down. The problems of the general Cournot model teach a lesson to modellers: sometimes simple assumptions such as linearity generate atypical results.

Many Oligopolists

Let us return to the simpler game in which production costs are zero and demand is linear. For concreteness, we will use the particular inverse demand function

$$p(q) = 120 - q. \tag{9}$$

Using (13.9), the payoff function, (13.2), becomes

$$\pi_a = 120 q_a - q_a^2 - q_b q_a. \tag{10}$$

In Section 3.5, firms picked outputs of 40 apiece given demand function (13.9). This generated a price of 40. With n firms instead of two, the demand function is

$$p\left(\sum_{i=1}^{n} q_i\right) = 120 - \sum_{i=1}^{n} q_i, \tag{11}$$

and firm j's payoff function is

$$\pi_j = 120 q_j - q_j^2 - q_j \sum_{i \neq j} q_i. \tag{12}$$

Differentiating j's payoff function with respect to q_j yields

$$\frac{d\pi_j}{dq_j} = 120 - 2q_j - \sum_{i \neq j} q_i = 0. \tag{13}$$

The first step in finding the equilibrium is to guess that it is symmetric, so that $q_j = q_i, (i = 1, \ldots, n)$. This is an educated guess, since every player faces a first-order condition like (13.13). By symmetry, equation (v13) becomes $120 - (n + 1)q_j = 0$, so that

$$q_j = \frac{120}{n+1}. \tag{14}$$

$$q_j = \frac{120}{n+1}.$$

Consider several different values for n. If $n = 1$, then $q_j = 60$, the monopoly optimum; and if $n = 2$ then $q_j = 40$, the Cournot output found in Section 3.5. If $n = 5$, $q_j = 20$; and as n rises, individual output shrinks to zero. Moreover, the total output of $nq_j = \frac{120n}{n+1}$ gradually approaches 120, the competitive output, and the market price falls to zero, the marginal cost of production. As the number of firms increases, profits fall.

13.2 Prices as Strategies: Bertrand Equilibrium

We will use the same demand function, equation (13.9), which implies that if p is the lowest price, $q = 120 - p$. In the Cournot model, firms chose quantities but allowed the market price to vary freely; in the Bertrand model, they choose prices and sell as much as they can. The strategies for Apex and Brydox are p_a and p_b. The payoff function for Apex (and analogously for Brydox) is

$$\pi_a = \begin{cases} p_a(120 - p_a) & \text{if } p_a < p_b \\ \frac{p_a(120-p_a)}{2} & \text{if } p_a = p_b \\ 0 & \text{if } p_a > p_b \end{cases}$$

The Bertrand game has a unique Nash equilibrium: $p_a = p_b = 0$.

$b > 0$ or $p_b > p_a > 0$) In either of these cases the firm with the higher price could deviate to a price below its rival and increase its profits from zero to some positive value.

($p_a = p_b > 0$) In this case, Apex could deviate to a price ϵ less than Brydox and its profit would rise, because it would go from selling half the market quantity to selling all of it with an infinitesimal decline in profit per unit sale.

$b = 0$ or $p_b > p_a = 0$) In this case, the firm with the price of zero could move from zero profits to positive profits by increasing its price slightly while keeping it below the other firm's price.

Capacity Constraints: The Edgeworth Paradox

Let us start by altering the Bertrand model by constraining each firm to sell no more than $K = 70$ units.

Intensity rationing. *The customers that value the product most buy from the firm with the lower price.*

Proportional rationing. *Each customer has the same probability of being able to buy from the low-price firm.*

Figure 13.2 Rationing Rules (a) intensity rationing if K=70; (b) proportional rationing

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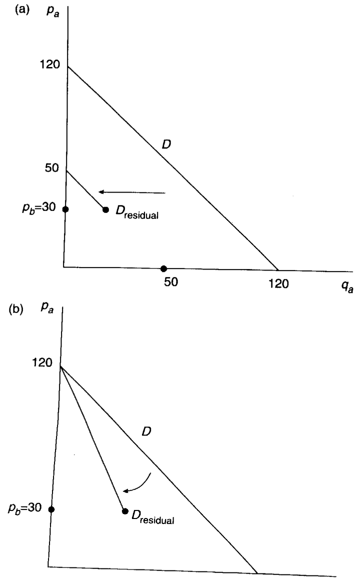

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The inverse demand function from equation (13.9) is $p = 120 - q$, and under intensity rationing, the K customers with the strongest demand buy from the low-price firm. Suppose that Brydox is the low-price firm, charging a price of 30 so that 90 consumers wish to buy from it. The residual demand facing Apex is then

$$q_a = 120 - p_a - K. \tag{15}$$

Under intensity rationing, the payoff functions are, given that $K = 70$,

$$\pi_a = \begin{cases} p_a \cdot Min\{120 - p_a, 70\} & \text{if } p_a < p_b \quad (a) \\ \frac{p_a(120-p_a)}{2} & \text{if } p_a = p_b \quad (b) \\ 0 & \text{if } p_a > p_b, p_b \geq 50 \quad (c) \\ p_a(120 - p_a - 70) & \text{if } p_a > p_b, p_b < 50 \quad (d) \end{cases} \tag{16}$$

Here is why equations (13.16c) and (13.16d) look the way they do. If Brydox has the lower price, all consumers will want to buy from Brydox if they buy at all, but only 70 will be able to. If Brydox's price is more than 50, then less than 70 will want to buy at all, and so 0 customers will be left for Apex– which is equation (13.16c). If Brydox's price is less than 50, then Brydox will sell 70 units, and the residual demand curve facing Apex is as in equation (13.15), yielding equation (13.16d).

Proportional rationing. *Each customer has the same probability of being able to buy from the low-price firm.*

Under proportional rationing, if $K = 70$ and 90 customers wanted to buy from Brydox, 2/9 $(= \frac{q(p_b)-K}{q(p_b)})$ of each type of customer will be forced to buy from Apex (for example, 2/9 of the type willing to pay 120). The residual demand curve facing Apex, shown in Figure 13.2b and equation (13.17), intercepts the price axis at 120, but slopes down at a rate three times as fast as market demand because there are only 2/9 as many remaining customers of each type.

$$q_a = (120 - p_a)\left(\frac{120 - p_b - K}{120 - p_b}\right) \tag{17}$$

Suppose that demand is linear, with the highest reservation price being $p = 100$ and the maximum market quantity $Q = 100$ at $p = 0$. Suppose also that there are two firms, Apex and Brydox, each having a constant marginal cost of 0 up to capacity of $Q = 80$ and infinity thereafter. We will assume intensity rationing of buyers.

Note that industry capacity of 160 exceeds market demand of 100 if price equals marginal cost. Note also that the monopoly price is 50, which with quantity of 50 yields industry profit of 2500. But what will be the equilibrium?

Prices of $(p_a = 0, p_b = 0)$ are not an equilibrium. Apex's profit would be zero in that strategy combination. If Apex increased its price to 5, what would happen? Brydox would immediately sell $Q = 80$, and to the most intense 80 percent of buyers. Apex would be left with all the buyers between $p = 20$ and $p = 5$ on the demand curve, for $Q_a = 15$ and profit of $\pi_a = (5)(15) = 75$. So deviation by Apex is profitable. (Of course, $p = 5$ is not necessarily the most profitable deviation– but we do not need to check that; I looked for an *easy* deviation.)

Equal prices of (p_a, p_b) with $p_a = p_b > 0$ are not an equilibrium. Even if the price is close to 0, Apex would sell at most 50 units as its half of the market, which is less than its capacity of 80. Apex could deviate to just below p_b and have a discontinuous jump in sales for an increase in profit, just as in the basic Bertrand game.

Unequal prices of (p_a, p_b) are not an equilibrium. Without loss of generality, suppose $p_a > p_b$. So long as p_b is less than the monopoly price of 50, Brydox would deviate to a new price even close to but not exceeding p_a. And this is not *just* the open-set problem. Once Brydox is close enough to Apex, Apex would deviate by jumping to a price just below Brydox.

If capacities are small enough, the Edgeworth Paradox also disappears, but so does the Bertrand Paradox. Suppose each firm has a capacity of 20. They each will choose to sell at a price of 60, in which case they will each sell 20 units, their entire capacities. Apex will have a payoff of 1200. If Apex deviates to a lower price, it will not sell any more, so that would be unprofitable. If Apex deviates to a higher price, it will sell fewer, and since the monopoly price is 50, its profit will be lower; note that a price of 61 and a quantity of 19 yields profits of 1159, for example.

A mixed strategy equilibrium does exist, calculated using intensity rationing by Levitan & Shubik (1972) and analyzed in Dasgupta & Maskin (1986b). Expected profits are positive, because the firms charge positive prices. Under proportional rationing, as under intensity rationing, profits are positive in equilibrium, but the high-price firm does better with proportional rationing. The high-price firm would do best with **inverse-intensity rationing**, under which the customers with the least intense demand are served at the low-price firm, leaving the ones willing to pay more at the mercy of the high-price firm.

Product Differentiation

If customers have brand loyalty or poor price information, the equilibrium is different and the demand curves facing Apex and Brydox might be

$$q_a = 24 - 2p_a + p_b \tag{18}$$

and

$$q_b = 24 - 2p_b + p_a. \tag{19}$$

The greater the difference in the coefficients on prices in demand curves like these, the less substitutable are the products. As with standard demand curves like (13.9), we have made implicit assumptions about the extreme points of (13.18) and (13.19). These equations only apply if the quantities demanded turn out to be nonnegative, and we might also want to restrict them to prices below some ceiling, since otherwise the demand facing one firm becomes infinite as the other's price rises to infinity. With those restrictions, the payoffs are

$$\pi_a = p_a(24 - 2p_a + p_b) \tag{20}$$

and

$$\pi_b = p_b(24 - 2p_b + p_a). \tag{21}$$

Maximizing Apex's payoff, we obtain the first-order condition

$$\frac{d\pi_a}{dp_a} = 24 - 4p_a + p_b = 0, \tag{22}$$

and the reaction function

$$p_a = 6 + p_b/4. \tag{23}$$

Since Brydox has a parallel first-order condition, the equilibrium occurs where $p_a = p_b = 8$. The quantity each firm produces is 16, which is below the 24 each would produce at prices of zero. Figure 13.3 shows that the reaction functions intersect. Apex's demand curve has the elasticity

$$\left(\frac{\partial q_a}{\partial p_a}\right) \cdot \left(\frac{p_a}{q_a}\right) = -2\left(\frac{p_a}{q_a}\right), \tag{24}$$

which is finite even when $p_a = p_b$, unlike the case of the standard Bertrand model.

Figure 13.3 Bertrand Reaction Functions with Differentiated Products

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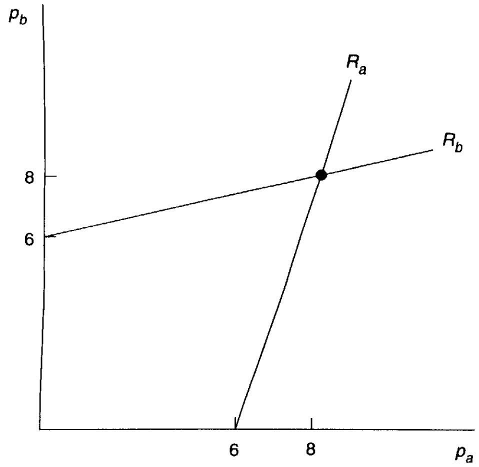


Cournot Equilibrium with Differentiated Products

We can also work out the Cournot equilibrium for demand functions (13.18) and (13.19), but product differentiation does not affect it by much. Start by expressing the price in terms of quantities alone, obtaining

$$p_a = 12 - \frac{1}{2}q_a + \frac{1}{2}p_b \tag{25}$$

and

$$p_b = 12 - \frac{1}{2}q_b + \frac{1}{2}p_a. \tag{26}$$

After substituting from (13.26) into (13.25) and solving for p_a, we obtain

$$p_a = 24 - 2q_a/3 - q_b/3. \tag{27}$$

The first-order condition for Apex's maximization problem is

$$\frac{d\pi_a}{dq_a} = 24 - 4q_a/3 - q_b/3 = 0, \tag{28}$$

which gives rise to the reaction function

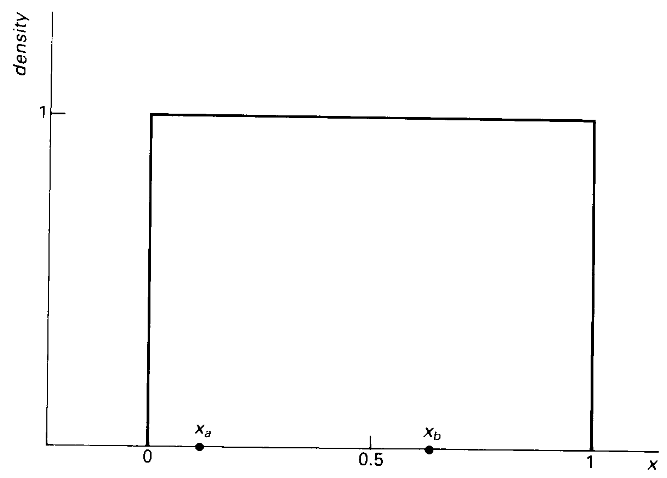
$$q_a = 18 - q_b/4. \tag{29}$$

We can guess that $q_a = q_b$. It follows from (31) that $q_a = 14.4$ and the market price is 9.6. On checking, you would find this to indeed be a Nash equilibrium.

13.3 Location Models

Figure 13.4: Location Models

Figure 12.4 Location models



“The Hotelling Pricing Game”

(Hotelling [1929])

Players
Sellers Apex and Brydox, located at x_a and x_b, where $x_a < x_b$, and a continuum of buyers indexed by location $x \in [0, 1]$.

The Order of Play
(1) The sellers simultaneously choose prices p_a and p_b.
(2) Each buyer chooses a seller.

Payoffs

Demand is uniformly distributed on the interval [0,1] with a density equal to one (think of each consumer as buying one unit). Production costs are zero. Each consumer always buys, so his problem is to minimize the sum of the price plus the linear transport cost, which is θ per unit distance travelled.

$$\pi_{buyer\ at\ x} = -Min\{\theta|x_a - x| + p_a,\ \theta|x_b - x| + p_b\}. \qquad (30)$$

$$\pi_a = \begin{cases} 0 & \text{if } p_a - p_b \text{ ¿ } \theta(x_b - x_a) \\ & \text{(Brydox captures entire market)} \\ & \\ p_a & \text{if } p_b - p_a > \theta(x_b - x_a) \\ & \text{(Apex captures entire market)} \\ & \\ p_a(\frac{1}{2\theta}[(p_b - p_a) + \theta(x_a + x_b)]) & \text{otherwise (the market is divided)} \end{cases}$$

The payoffs result from buyer behavior. A buyer's utility depends on the price he pays and the distance he travels. Price aside, Apex is most attractive to the customer at $x = 0$ ("Customer 0") and least attractive to the customer at $x = 1$ ("Customer 1"). Customer 0 will buy from Apex so long as

$$\theta x_a + p_a < \theta x_b + p_b, \qquad (32)$$

which implies that

$$p_a - p_b < \theta(x_b - x_a), \qquad (33)$$

which yields payoff (13.31a). Customer 1 will buy from Brydox if

$$\theta(1 - x_a) + p_a > \theta(1 - x_b) + p_b, \qquad (34)$$

which implies that

$$p_b - p_a < \theta(x_b - x_a), \qquad (35)$$

which yields payoff (13.31b).

$$p_a - p_b < \theta(x_b - x_a),$$
$$p_b - p_a < \theta(x_b - x_a),$$

Very likely, inequalities (13.33) and (13.35) are both satisfied, in which case Customer 0 goes to Apex and Customer 1 goes to Brydox. This is the case represented by payoff (13.31c), and the next task is to find the location of Customer x^*, defined as the customer who is at the boundary between the two markets, indifferent between Apex and Brydox. First, notice that if Apex attracts Customer x_b, he also attracts all $x > x_b$, because beyond x_b the customers' distances from both sellers increase at the same rate. So we know that if there is an indifferent consumer he is between x_a and x_b. Knowing this, (13.30) tells us that

$$\theta(x^* - x_a) + p_a = \theta(x_b - x^*) + p_b, \tag{36}$$

so that

$$p_b - p_a = \theta(2x^* - x_a - x_b), \tag{37}$$

and

$$x^* = \frac{1}{2\theta}\left[(p_b - p_a) + \theta(x_a + x_b)\right]. \tag{38}$$

Keep in mind that equation (13.38) is valid only if there really does exist a consumer who is indifferent– if such a consumer does not exist, equation (13.38) will generates a number for x^*, but that number is meaningless.

Since Apex keeps all the customers between 0 and x^*, equation (13.38) is the demand function facing Apex so long as he does not set his price so far above Brydox's that he loses even

Customer 0. The demand facing Brydox equals $(1-x^*)$. Note that if $p_b = p_a$, then from (13.38), $x^* = \frac{x_a+x_b}{2}$, independent of θ, which is just what we would expect. Demand is linear in the prices of both firms, and looks similar to demand curves (13.18) and (13.19), which were used in Section 13.2 for the Bertrand game with differentiated products.

Now that we have found the demand functions, the Nash equilibrium can be calculated in the same way as in Section 13.2, by setting up the profit functions for each firm, differentiating with respect to the price of each, and solving the two first-order conditions for the two prices. If there exists an equilibrium in which the firms are willing to pick prices to satisfy inequalities (13.33) and (13.35), then it is

$$p_a = \frac{(2 + x_a + x_b)\theta}{3}, \quad p_b = \frac{(4 - x_a - x_b)\theta}{3}. \qquad (39)$$

From (13.39) one can see that Apex charges a higher price if a large x_a gives it more safe customers or a large x_b makes the number of contestable customers greater. The simplest case is when $x_a = 0$ and $x_b = 1$, when (13.39) tells us that both firms charge a price equal to θ. Profits are positive and increasing in the transportation cost.

We cannot rest satisfied with the neat equilibrium of equation (13.39), however, because the assumption that there exists an equilibrium in which the firms choose prices so as to split the market on each side of some boundary consumer x^* is often violated. Hotelling did not notice this, and fell into common trap of game theory situations. Economists are used to models in which the calculus approach gives an answer that is both the local optimum and the global optimum. In games like this one, however, the local optimum is not global, because of the discontinuity in the objective function. Vickrey (1964) and D'Aspremont, Gabszewicz & Thisse (1979) have shown that if x_a and x_b are close together, no pure-strategy equilibrium exists, for reasons similar to why none exists in the Bertrand model with capacity constraints. If both firms charge non-random prices, neither would deviate to a slightly different price, but one might deviate to a much lower price that would capture every single customer. But if both firms charged that low price, each would deviate by raising his price slightly. It turns out that if Apex and Brydox are located symmetrically around the center of the interval, so $x_a \geq 0.25$ and $x_b \leq 0.75$, no pure- strategy equilibrium exists (although a mixed-strategy equilibrium does exist, as Dasgupta & Maskin [1986b] show).

Example 1. Try $x_a = 0, x_b = .7$ and $\theta = .5$. Then equation (13.39) says $p_a = (2 + 0 + .7).5/3 = 0.45$ and $p_b = (4 - 0 - .7).5/3 = 0.55$. Equation (13.38) says that $x^* = \frac{1}{2*0.5}[(0.55 - 0.45) + 0.5(0 + .7)] = .45$.

In Example 1, there is a pure strategy equilibrium and the equations generated sensible numbers given the parameters we chose. But it is not enough to calculate just one numerical example.

Example 2. Try $x_a = .9, x_b = .9$ and $\theta = .5$. Then equation (13.39) says $p_a = (2 + .9 + .9).5/3 \approx .63$ and $p_b = (4 - .9 - .9).5/3 \approx .37$.

Example 2 shows something odd happening. The equations generate numbers that seem innocuous until one realizes that if $p_a = 0.63$ and $p_b = 0.37$, Brydox will capture the entire market! The result is nonsense, because equation (13.39)'s derivation relied on the assumption that $x_a < x_b$, which is false in this example.

Example 3. Try $x_a = .7, x_b = .9$ and $\theta = .5$. Then equation (13.39) says that $p_a = (2 + .7 + .9).5/3 = .6$ and $p_b = (4 - .7 - .9).5/3 = .4$. As for the split of the market, equation (13.38) says that $x^* = \frac{1}{2*.5}\left[(.4 - .6) + .5(.7 + .9)\right] = .6$.

If the market splits at $x^* = 0.6$ but $x_a = 0.7$ and $x_b = 0.9$, the result violates our implicit assumption that the players split the market. Equation (13.38) is based on the premise that there does exist some indifferent consumer, and when that is a false premise, as under the parameters of Example 3, equation (13.38) will still spit out a value of x^*, but the value will not mean anything. And in fact the consumer at $x = 0.6$ is not really indifferent between Apex and Brydox. 0He could buy from Apex at a total cost of 0.6 + .1(0.5) = 0.65 or from Brydox, at a total cost of 0.4 + 0.3 (0.5) = 0.55. In fact, there exists no consumer who strictly prefers Apex . Even Apex's 'home' consumer at $x = 0.7$ would have a total cost of buying from Brydox of $0.4 + 0.5(0.9 - 0.7) = 0.5$ and would prefer Brydox. Similarly, the consumer at $x = 0$ would have a total cost of buying from Brydox of $0.4 + 0.5(0.9 - 0) = 0.85$, compared to a cost from Apex of $0.6 + 0.5(0.7 - 0) = 0.95$, and he, too, would prefer Brydox.

"The Hotelling Location Game"

(Hotelling [1929])

Players

n Sellers.

The Order of Play

The sellers simultaneously choose locations $x_i \in [0,1]$.

Payoffs

Consumers are distributed along the interval [0,1] with a uniform density equal to one. The price equals one, and production costs are zero. The sellers are ordered by their location so $x_1 \leq x_2 \leq \ldots \leq x_n$, $x_0 \equiv 0$ and $x_{n+1} \equiv 1$. Seller i attracts half the customers from the gaps on each side of him, so that his payoff is

$$\pi_1 = x_1 + \frac{x_2 - x_1}{2}, \tag{40}$$

$$\pi_n = \frac{x_n - x_{n-1}}{2} + 1 - x_n, \tag{41}$$

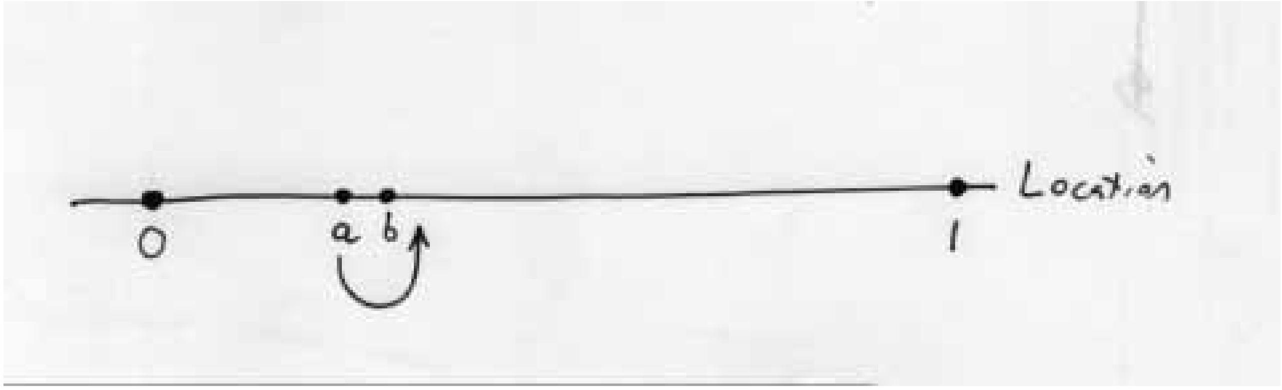
or, for $i = 2, \ldots n-1$,

$$\pi_i = \frac{x_i - x_{i-1}}{2} + \frac{x_{i+1} - x_i}{2}. \tag{42}$$

With **one seller**, the location does not matter in this model, since the customers are captive. If price were a choice variable and demand were elastic, we would expect the monopolist to locate at $x = 0.5$.

With **two sellers**, both firms locate at $x = 0.5$, regardless of whether or not demand is elastic. This is a stable Nash equilibrium, as can be seen by inspecting Figure 13.4 and imagining best responses to each other's location. The best response is always to locate ε closer to the center of the interval than one's rival. When both firms do this, they end up splitting the market since both of them end up exactly at the center.

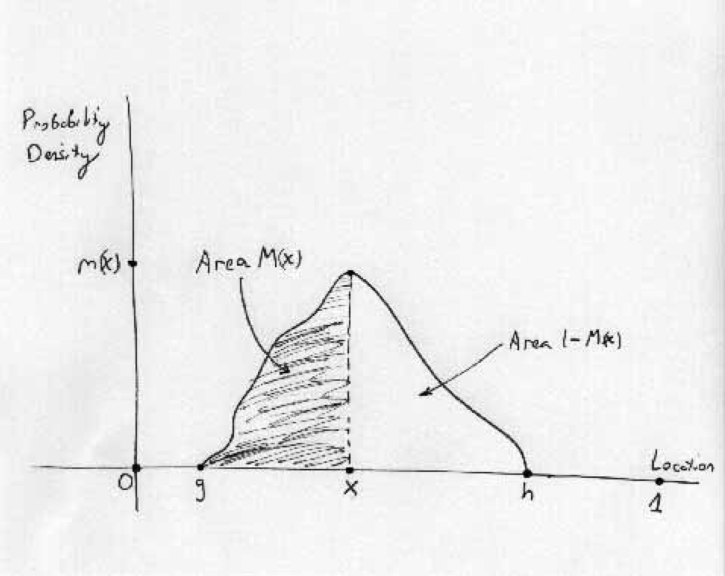
Figure 13.5 Nonexistence of Pure Strategies with Three Players



With **three sellers** the model does not have a Nash equilibrium in pure strategies.

Strangely enough, three is a special number. With **more than three sellers**, an equilibrium in pure strategies does exist if the consumers are uniformly distributed.

Figure 13.6 The Equilibrium Mixed Strategy Density in the Three-Player Location Game



Suppose all three players use the same mixing density, with $m(x)$ the probability density for location x, and positive density on the support $[g, h]$, as depicted in Figure 13.6.

We will need the density for the distribution of the minimum of the locations of Firms 2 and 3. Firm 2 has location x with density $m(x)$, and Firm 3's location is greater than that with probability $1 - M(x)$, letting M denote the cumulative distribution, so the density for Firm 2 having location x and it being smaller is $m(x)[1 - M(x)]$. The density for either Firm 2 or Firm 3 choosing x and it being smaller than the other firm's location is then $2m(x)[1 - M(x)]$.

We will also need the density for the distribution of the maximum of the locations of Firms 2 and 3. Firm 2 has location x with density $m(x)$, and Firm 3's location is less than that with probability $M(x)$, so the density for Firm 2 having location x and it being smaller is $m(x)M(x)$. The density for either Firm 2 or Firm 3 choosing x and it being smaller than the other firm's location is then $2m(x)M(x)$.

If Player 1 chooses $x = g$, then his expected payoff is

$$\pi_1(x_1 = g) = g + \int_g^h 2m(x)[1 - M(x)] \left(\frac{x - g}{2} \right) dx, \tag{43}$$

where g is the safe set of customers to his left, $2m(x)[1 - M(x)]$ is the density for x being the next biggest firm location, and $\frac{x-g}{2}$ is Firm 1's share of the customers between his own location of g and the next biggest location of x.

If Player 1 chooses $x = h$, then his expected payoff is, similarly,

$$\pi_1(x_1 = h) = h + \int_g^h 2m(x)M(x) \left(\frac{h - x}{2} \right) dx \tag{44}$$

In a mixed strategy equilibrium, Player 1's payoffs from these two pure strategies must be equal, and they are also equal to his payoff from a location of 0.5, which we can plausibly guess is in the support of his mixing distribution. Going on from this point, the algebra and calculus start to become fierce. Shaked (1982) has computed the symmetric mixing probability density $m(x)$ to be

$$m(x) = \begin{cases} 2 & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

*13.4 Comparative Statics and Supermodular Games

Let us use a differentiated Bertrand game as an example. Suppose there are N firms, and for firm n the demand curve is

$$Q_n = Max\{\alpha - \beta_n p_n + \sum_{m \neq n} \gamma_m p_m, 0\}, \tag{46}$$

with $\alpha \in (0, \infty)$, $\beta_n \in (0, \infty)$, and $\gamma_n \in (0, \infty)$ for all n. Assume that the effect of p_n on firm n's sales is larger than the effect of the other firms' prices, so that

$$\beta_n > \sum_{m \neq n} \gamma_m. \tag{47}$$

Let firm n have constant marginal cost κc_n, where $\kappa \in \{1, 2\}$ and $c_n \in (0, \infty)$, and let us assume that firm n's costs are low enough that it does operate in equilibrium. The shift variable κ represents the effect of the political regime on costs. The payoff function for firm n is

$$\pi_n = (p_n - \kappa c_n)(\alpha - \beta_n p_n + \sum_{m \neq n} \gamma_m p_m). \tag{48}$$

Firms choose prices simultaneously.

Does this game have an equilibrium? Does it have several equilibria? What happens to the equilibrium price if a parameter such as c_n or κ changes? These are difficult questions because if c_n increases, the immediate effect is to change firm n's price, but the other firms will react to the price change, which in turn will affect n's price. Moreover, this is not a symmetric game— the costs and demand curves differ from firm to firm, which could make algebraic solution of the Nash equilibrium quite messy. It is not even clear whether the equilibrium is unique.

The Implicit Function Theorem

The implicit-function theorem says that if $f(x,y)=0$, then

$$\frac{\partial x}{\partial y} = -\left(\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}\right). \tag{49}$$

This is especially useful if x is a choice variable and y a parameter, because then the first-order condition takes the form $f(x,y) = 0$, and the second-order condition determines the sign of $\frac{\partial f}{\partial x}$. One only has to make certain that the solution is an interior solution, so the first- and second-order conditions are valid.

In the differentiated Bertrand game, equilibrium prices will lie inside the interval $(c_n, \overline{p})$ for some large number $\overline{p}$, because a price of c_n would yield zero profits, rather than the positive profits of a slightly higher price, and $\overline{p}$ can be chosen to yield zero quantity demanded and hence zero profits. The equilibrium or equilibria are, therefore, interior solutions, in which case in this well-behaved problem they satisfy the first-order condition,

$$\frac{\partial \pi_n}{\partial p_n} = \alpha - 2\beta_n p_n + \sum_{m \neq n} \gamma_m p_m + \kappa c_n \beta_n = 0, \tag{50}$$

and the second-order condition,

$$\frac{\partial^2 \pi_n}{\partial p_n^2} = -2\beta_n < 0. \tag{51}$$

We can apply the implicit function theorem by letting $\frac{\partial \pi_n(p_n,c_n)}{\partial p_n} = 0$ from equation (50) be our $f(x,y) = 0$ and using equation

(49). Then

$$\begin{aligned}\frac{\partial p_n}{\partial c_n} &= -\left(\frac{\frac{\partial^2 \pi_n}{\partial p_n \partial c_n}}{\frac{\partial^2 \pi_n}{\partial p_n^2}}\right) \\ &= -\left(\frac{\kappa\beta_n}{-2\beta_n}\right) \\ &= \frac{\kappa}{2}.\end{aligned}$$

Thus, an increase in firm n's individual cost parameter increases its price at a rate of $\kappa/2$.

Supermodularity

The second approach uses the idea of the supermodular game, an idea related to that of strategic complements. Suppose that there are N players in a game, subscripted by m and n, and that player n has a strategy consisting of k_n elements, subscripted by i and j, so his strategy is the vector $x_n = (x_{n1}, \ldots, x_{nk_n})$. Let his strategy set be S_n and his payoff function be $\pi_n(x_n, x_{-n}; \tau)$, where τ represents a fixed parameter. We say that the game is a **smooth supermodular game** if the following four conditions are satisfied:

(A1$'$) The strategy set is an interval in $\boldsymbol{R}^{kn}$:

$$S_n = [\underline{x_n}, \overline{x_n}]. \tag{52}$$

(A2$'$) π_n is twice continuously differentiable on S_n.

(A3$'$) (supermodularity) Increasing one component of player n's strategy does not decrease the net marginal benefit of any other component: for all n, and all i and j such that $1 \leq i < j \leq k_n$,

$$\frac{\partial^2 \pi_n}{\partial x_{ni} \partial x_{nj}} \geq 0. \tag{53}$$

(A4$'$) (increasing differences in one's own and other strategies) Increasing one component of n's strategy does not decrease the net marginal benefit of increasing any component of player m's strategy: for all $n \neq m$, and all i and j such that $1 \leq i \leq k_n$ and $1 \leq j \leq k_m$,

$$\frac{\partial^2 \pi_n}{\partial x_{ni} \partial x_{mj}} \geq 0. \tag{54}$$

In addition, we will be able to talk about the comparative statics of smooth supermodular games if a fifth condition is satisfied, the increasing differences condition, (A5$'$).

(A5$'$) (increasing differences in one's own strategies and parameters) Increasing parameter c does not decrease the net marginal benefit to player n of any component of his ownstrategy: for all n, and all i such that $1 \leq i \leq k_n$,

$$\frac{\partial^2 \pi_n}{\partial x_{ni} \partial \tau} \geq 0. \tag{55}$$

Is the differentiated Bertrand game supermodular? The strategy set can be restricted to $[c_n, \overline{p}]$ for player n, so (A1$'$) is satisfied. π_n is twice continuously differentiable on the interval$[c_n, \overline{p}]$, so (A2$'$) is satisfied. A player's strategy has just one component, p_n, so (A3$'$) is immediately satisfied. The following inequality is true,

$$\frac{\partial^2 \pi_n}{\partial p_n \partial p_m} = \gamma_m > 0, \qquad (56)$$

so (A4$'$) is satisfied. And it is also true that

$$\frac{\partial^2 \pi_n}{\partial p_n \partial c_n} = \kappa \beta_n > 0, \qquad (57)$$

so (A5$'$) is satisfied for c_n.

From equation (13.50), $\frac{\partial \pi_n}{\partial p_n}$ is increasing in κ, so $\pi_n(p_n, p_{-n}, \kappa) - \pi_n(p'_n, p_{-n}, \kappa)$ is nondecreasing in κ for $p_n > p'_n$, and (A5) is satisfied for κ.

Thus, all the assumptions are satisfied. This being the case, a number of theorems can be applied. Two of them are Theorems 13.1 and 13.2.

Theorem 13.1
If the game is supermodular, there exists a largest and a smallest Nash equilibrium in pure strategies.

Theorem 13.2
If the game is supermodular and assumption (A5) or (A5$'$) is satisfied, then the largest and smallest equilibrium are nondecreasing functions of the parameter τ*.*

Applying Theorems 13.1 and 13.2 yields the following results for the differentiated Bertrand game:

(1) There exists a largest and a smallest Nash equilibrium in pure strategies (Theorem 13.1).

(2) The largest and smallest equilibrium prices for firm n are nondecreasing functions of the cost parameters c_n and κ (Theorem 13.2).

Theorem 13.2 is also useful in proving that the equilibrium here is, in fact, unique— the largest and smallest equilibrium are one and the same. Since

$$\frac{\partial^2 \pi_n}{\partial p_n p_m} = \gamma_m, \tag{58}$$

it will be true that

$$-\left(\frac{\partial^2 \pi_n}{\partial p_n^2}\right) > \sum_{m \neq n} \frac{\partial^2 \pi_n}{\partial p_n p_m}. \tag{59}$$

Condition (13.59) is what is commonly called a **dominant-diagonal condition**. It says that direct effects on profits are more important than all the indirect effects, so if one expresses the second derivatives in matrix form, the main diagonal would have the largest elements. For a three-firm case that matrix would be

$$\begin{bmatrix} \frac{\partial^2 \pi_1}{\partial p_1^2} & \frac{\partial^2 \pi_1}{\partial p_1 \partial p_2} & \frac{\partial^2 \pi_1}{\partial p_1 \partial p_3} \\ \frac{\partial^2 \pi_2}{\partial p_2 \partial p_1} & \frac{\partial^2 \pi_2}{\partial p_2^2} & \frac{\partial^2 \pi_2}{\partial p_2 \partial p_3} \\ \frac{\partial^2 \pi_3}{\partial p_3 \partial p_1} & \frac{\partial^2 \pi_3}{\partial p_3 \partial p_2} & \frac{\partial^2 \pi_3}{\partial p_3^2} \end{bmatrix} \tag{60}$$

Suppose there were two equilibrium price combinations, p and $\hat{p}$. Theorem 13.1 says that the largest and smallest equilibria can be ranked, so for every strategy in the strategy combination, it would be true that $\hat{p} \geq p$. But because the first-order condition applies at both equilibria, we know that

$$\frac{\partial \pi_n(p)}{\partial p_n} - \frac{\partial \pi_n(\hat{p})}{\partial p_n} = 0. \tag{61}$$

One can rewrite equation (13.61) differently. Starting at equilibrium p and moving to $\hat{p}$, the first derivative would change as all the components of p changed. If we use t to index the slow changes in the components, we can write these changes as

$$\int_0^1 \left\{ \left((\hat{p_n} - p_n) \cdot \frac{\partial^2 \pi_n[t\hat{p} + (1-t)p]}{\partial p_n^2} \right) + \left(\sum_{m \neq n} (\hat{p_m} - p_m) \cdot \frac{\partial^2 \pi_n[t\hat{p} + (1-t)p]}{\partial p_n \partial p_m} \right) \right\} dt \tag{62}$$

Expression (13.62) equals expression (13.61) . But from equation (13.59), expression (13.62) must be negative, and equation (13.61) equals zero. This is a contradiction, so there cannot really be two different equilibria. The biggest and smallest equilibria are one and the same, and the equilibrium is unique.

13.5 Durable Monopoly

“Durable Monopoly”

Players
A buyer and a seller.

The Order of Play
(1) The seller picks the first-period price, p_1.
(2) The buyer buys quantity q_1 and consumes service flow q_1.
(3) The seller picks the second-period price, p_2.
(4) The buyer buys additional quantity q_2 and consumes service flow $(q_1 + q_2)$.

Payoffs
Production cost is zero and there is no discounting. The seller's payoff is his revenue, and the buyer's payoff is the sum across periods of his benefits from consumption minus his expenditure. His benefits arise from his being willing to pay as much as

$$B(q_t) = 60 - \frac{q_t}{2} \tag{63}$$

for the marginal unit service flow consumed in period t, as shown in Figure 13.7. The payoffs are therefore

$$\pi_{seller} = q_1 p_1 + q_2 p_2 \tag{64}$$

and

$$
\begin{aligned}
\pi_{buyer} &= [consumer\ surplus_1] + [consumer\ surplus_2] \\
&= [total\ benefit_1 - expenditure_1] + [total\ benefit_2 - expenditure_2] \\
&= \left[\tfrac{(60-B(q_1))q_1}{2} + B(q_1)q_1 - p_1 q_1\right] \\
&\quad + \left[\tfrac{60-B(q_1+q_2)}{2}(q_1+q_2) + B(q_1+q_2)(q_1+q_2) - p_2 q_2\right]
\end{aligned}
\tag{65}
$$

In the first period, the marginal unit consumed was the q_1-th. In the second period, it will be the (q_1+q_2)-th. The residual demand curve after the first period's purchases is shown in Figure 13.7b. It is a demand curve very much like the demand curve resulting from intensity rationing in the capacity-constrained Bertrand game of Section 13.2, as shown in Figure 13.2a. The most intense portion of the buyer's demand, up to q_1 units, has already been satisfied, and what is left begins with a marginal benefit of $B(q_1)$, and falls at the same slope as the original marginal benefit curve. The equation for the residual demand is therefore, using equation (66),

$$p_2 = B(q_1) - \frac{q_2}{2} = 60 - \frac{(q_1)}{2} - \frac{(q_2)}{2}. \tag{66}$$

Solving for the monopoly quantity, q_2^*, the seller maximizes $q_2 p_2$, solving the problem

$$\overset{Maximize}{q_2} \quad q_2\left(60 - \frac{q_1+q_2}{2}\right), \tag{67}$$

which generates the first-order condition

$$60 - q_2 - q_1/2 = 0, \tag{68}$$

so that

$$q_2^* = 60 - q_1/2. \tag{69}$$

From equations (13.63) and (13.69), it can be seen that $p_2^* = 30 - q_1/4$.

We must now find q_1^*. In period one, the buyer looks ahead to the possibility of buying in period two at a lower price. Buying in the first period has two benefits: consumption of the service flow in the first period and consumption of the service flow in the second period. The price he would pay for a unit in period one cannot exceed the marginal benefit from the first-period service flow in period one plus the foreseen value of p_2, which from (13.69) is $30 - q_1/4$. If the seller chooses to sell q_1 in the first period, therefore, he can do so at the price

$$\begin{aligned} p_1(q_1) &= B(q_1) + p_2 \\ &= (60 - q_1/2) + (30 - q_1/4), \\ &= 90 - \tfrac{3}{4}q_1. \end{aligned} \tag{70}$$

Knowing that in the second period he will choose q_2 according to (13.69), the seller combines (13.69) with (13.70) to give the maximand in the problem of choosing q_1 to maximize profit over the two periods, which is

$$\begin{aligned} (p_1q_1 + p_2q_2) &= (90 - \tfrac{3}{4}q_1)q_1 + (30 - q_1/4)(60 - q_1/2) \\ &= 1800 + 60q_1 - \tfrac{5}{8}q_1^2, \end{aligned} \tag{71}$$

which has the first-order condition

$$60 - \frac{5}{4}q_1 = 0, \tag{72}$$

so that

$$q_1^* = 48 \tag{73}$$

and, making use of (13.70), $p_1^* = 54$.

It follows from (13.69) that $q_2^* = 36$ and $p_2 = 18$. The seller's profits over the two periods are $\pi_s = 3,240$ $(= 54(48) + 18(36))$.

A *competitive market* bids down the price to the marginal cost of zero. Then, $p_1 = 0$ and $q_1 = 120$ from (13.63), and profits equal zero.

If the monopolist *rents* instead of selling, then equation (13.63) is like an ordinary demand equation, because the monopolist is effectively selling the good's services separately each period. He could rent a quantity of 60 each period at a rental fee of 30 and his profits would sum to $\pi_s = 3,600$. That is higher than 3,240, so profits are higher from renting than from selling outright.

If the monopolist can *commit to not producing in the second period*, he will do just as well as the monopolist who rents, since he can sell a quantity of 60 at a price of 60, the sum of the rents for the two periods.

If the modeller ignored sequential rationality and simply looked for the Nash equilibrium that maximized the payoff of the seller by his choice of p_1 and p_2, he would come to the commitment result. An example of such an equilibrium is ($p_1 =$ 60, $p_2 = 200$, *Buyer purchases according to* $q_1 = 120 - p_1$, *and* $q_2 = 0$). This is Nash because neither player has incentive to deviate given the other's strategy, but it fails to be subgame perfect, because the seller should realize that if he deviates and chooses a lower price once the second period is reached, the buyer will respond by deviating from $q_2 = 0$ and will buy more units.

With more than two periods, the difficulties of the durable-goods monopolist become even more striking. In an infinite-period model without discounting, if the marginal cost of production is zero, the equilibrium price for outright sale instead of renting is constant— at zero!