# Extra Questions for Games and Information, 3rd Edition

September 12, 1999

Eric Rasmusen

*Abstract*

These are extra questions for G601, a PhD course in game theory, information economics, and modelling.

Indiana University, Kelley School of Business,BU 456, 1309 E 10th Street, Bloomington, Indiana, 47405-1701. Office: (812) 855-9219. Fax: 812-855-3354. Email: Erasmuse@indiana.edu. Web: Php.indiana.edu/∼erasmuse.

# 1. The Rules of the Game

1.1x. Find the Nash equilibria of the following game, illustrated in Table 1.1. Can any of them be reached by iterated dominance?

**Table 1.1 A Midterm Game**

| | | **Column** | | |
|---|---|---|---|---|
| | | *Left* | *Middle* | *Right* |
| | *Up* | 10,10 | 0, 0 | −1, 15 |
| **Row:** | *Sideways* | −12, 1 | **8,8** | −1, −1 |
| | *Down* | **15,1** | 8,−1 | 0, 0 |

*Payoffs to: (Row, Column).*

ANSWER. The Nash equilibria are boldfaced. DL can be reached by iterated dominance. Iterate in the order: UP, SIDEWAYS, MIDDLE, RIGHT.

1.2x. What is a Nash equilibrium in Table 1.2, if it is a simultaneous-move game?
(a) $Flavor, Flavor$
@ (b) $Flavor, Texture$
@ (c) $Texture, Flavor$
(d) $Texture, Texture$

**Table 1.2: Flavor and Texture I**

| | | **Brydox** | |
|---|---|---|---|
| | | *Flavor* | *Texture* |
| **Apex:** | *Flavor* | -2,0 | 0,1 |
| | *Texture* | -1,-1 | 0,-2 |

*Payoffs to: (Apex, Brydox).*

1.3x. Using iterated dominance, what is the equilibrium in Table 1, if it is a simultaneous-move game?
(a) *Flavor, Flavor*
(b) *Flavor, Texture*
@ (c) *Texture, Flavor*
(d) *Texture, Texture*
(e) There is no such equilibrium

1.4x. What is the Nash equilibrium in Table 1.3, if it is a simultaneous-move game?
(a) *Flavor, Flavor*
@ (b) *Flavor, Texture*
@(c) *Texture, Flavor*
(d) *Texture, Texture*
(e) There is no such equilibrium

**Table 1.3: Flavor and Texture II**

| | | **Brydox** | |
|---|---|---|---|
| | | *Flavor* | *Texture* |
| **Apex:** | *Flavor* | 20,7 | 28,8 |
| | *Texture* | 25,5 | 28,3 |

*Payoffs to: (Apex, Brydox).*

1.5x. What is the Nash equilibrium in Table 1.4, if it is a simultaneous-move game?
(a) *Flavor, Flavor*
@ (b) *Flavor, Texture*
@ (c) *Texture, Flavor*
(d) *Texture, Texture*

**Table 1.4: Flavor and Texture III**

| | | **Brydox** | |
|---|---|---|---|
| | | *Flavor* | *Texture* |
| **Apex:** | *Flavor* | 120, 107 | 128,108 |
| | *Texture* | 125, 105 | 128,103 |

*Payoffs to: (Apex, Brydox).*

1.6x. Using iterated dominance, what is the equilibrium in Table 1.4, if it is a simultaneous-move game?
(a) $Flavor, Flavor$
(b) $Flavor, Texture$
@ (c) $Texture, Flavor$
(d) $Texture, Texture$
(e) There is no such equilibrium

1.7x. The following is the payoff matrix for
(a.) a version of the Battle of the Sexes.
(b.) a version of the Prisoner's Dilemma.
(c.) a version of Pure Coordination.
(d.) a version of the Legal Settlement Game.
@ (e.) none of the above.

| | | COL | |
|---|---|---|---|
| | | A | B |
| ROW | A | 3,3 | 0,1 |
| | B | 5,0 | -1,-1 |

1.8x. The following game has how many pure strategy Nash equilibria?
(a.) zero
(b.) one
@(c.) two
(d.) three

| | | COL | |
|---|---|---|---|
| | | Up | Down |
| ROW | Right | 1,250 | -1, 22 |
| | Left | 1,1 | 0,0 |

1.9x. The problem of deciding whether to adopt IBM or HP computers by two offices in a company is like
(a.) the prisoner's dilemma
(b.) the welfare game
@(c.) The battle of the sexes.

**1.10x. Find the pure-strategy Nash equilibria of the following game.**

**Table 1.7 A Final Game**

| | | **Column** | | |
|---|---|---|---|---|
| | | *Left* | *Middle* | *Right* |
| | *Up* | 9,2 | 0, 0 | 0, 9 |
| **Row:** | *Sideways* | 7, 1 | 8,8 | 5,5 |
| | *Down* | 8,4 | 8,2 | 0, 0 |

*Payoffs to: (Row, Column).*

ANSWER: (*Sideways, Middle*) is the only Nash equilibrium. It happens to be weak, but it is still unique.

1.11x. The large Wall Street investment banks have recently agreed not to make campaign contributions to state treasurers, which up till now has been a common practice. What was the game in the past, and why can the banks expect this agreement to hold fast?

ANSWER. This game was like a Prisoner's Dilemma. Suppose there are two investment banks. The one who made the largest contribution would get the state's bond issuing business, but all profits would be eaten up in contributions if both made contributions. Both would be better off if they refrained from making any contributions. These are the payoffs of a Prisoner's Dilemma.

This is a repeated game, however, and if one bank deviates by making contributions, other banks will resume making contributions, so the gain will be temporary. Moreover, if the contributions are public, the other banks can in fact respond immediately, before the underwriter for the bond issue is decided upon, so the deviating bank does not even get a temporary advantage.

Note that less efficient banks, which probably includes the Small regional ones like the Stephens bank in Arkansas, would prefer the old system, since they have a comparative advantage in corruption .

1.12x. Identify any dominated strategies and any Nash equilibria in pure strategies in the following game.

**Table 1.8: A Game for the 1997 Final**

| | | **Column** | | |
|---|---|---|---|---|
| | | *Left* | **Middle** | *Right* |
| | *Up* | **1,4** | $5,-1$ | $0,1$ |
| **Row:** | **Sideways** | $-1,0$ | -2,-2 | $-3,4$ |
| | *Down* | $0,3$ | $9,-1$ | $5,0$ |

*Payoffs to: (Row, Column).*

ANSWER: Middle and Sideways are dominated. *Up, Left* is Nash. Note that I did not ask about iterated dominance, which is a separate issue entirely. Using iterated dominance will not tell you what strategies are dominated or give you a complete set of Nash equilibria.

**1.13x: Timmy and Scarface.** Players Timmy and Scarface are caught in a game like the "Prisoner's Dilemma" except that Scarface already has a criminal record, so he will always get a prison term at least 5 years greater than Timmy, regardless of who finks and who denies. Construct an outcome matrix (with Scarface as Row) and find the Nash equilibrium for this game. (Note: There are at least two games that reasonably fit this story.)

*Answer.* The story is too vague to tell us exactly which game Scarface and Timmy are playing, so I will give two possibilities. Table A.2 is constructed by just subtracting 5 from each of Scarface's payoffs in the original "Prisoner's Dilemma" in Table 1.1. In equilibrium, Scarface denies and Timmy confesses.

**Table A.2 "Scarface I"**

| | | **Timmy** | | |
|---|---|---|---|---|
| | | *Deny* | | *Confess* |
| | *Deny* | $-6, -1$ | $\rightarrow$ | **-15,0** |
| **Scarface:** | | $\uparrow$ | | $\uparrow$ |
| | *Confess* | $-15, -10$ | $\leftarrow$ | $-13, -8$ |

*Payoffs to: (Scarface, Timmy).*

Table A.2 is a little far-fetched, because it implies that when Scarface confesses, Timmy's denial increases *Scarface*'s punishment, as well as Timmy's. This is possible. Maybe the judge wants to punish Timmy more (for denying), but must always punish Scarface more than Timmy. But Table A.3 shows another game to fit the story, one which preserves the "Prisoner's Dilemma" property that a prisoner is treated more leniently for providing useful evidence. Here, (*Confess*,*Confess*) is the Nash equilibrium, even though *Confess* is not a dominant strategy for Scarface (he would *Deny* if he thought Timmy would go along with him).

**Table A.3 "Scarface II"**

| | | **Timmy** | | |
|---|---|---|---|---|
| | | *Deny* | | *Confess* |
| | *Deny* | $-6, -1$ | $\rightarrow$ | $-30, 0$ |
| **Scarface:** | | $\uparrow$ | | $\downarrow$ |
| | *Confess* | $-13, -8$ | $\rightarrow$ | **-20,-5** |

*Payoffs to: (Scarface, Timmy).*

1.14x. Suppose Row moves first, then Column, in the Prisoner's Dilemma. What are the possible actions? What are the possible strategies? Construct a normal form, showing the relationship between strategy profiles and payoffs.

Hint: The normal form is *not* a two-by-two matrix here.

ANSWER: The possible actions are $Confess$ and $Deny$ for each player.

For Column, the strategy set is:

$$\left\{ \begin{array}{l} (C|C,\ C|D), \\ (C|C,\ D|D), \\ (D|C,\ D|D), \\ (D|C,\ C|D) \end{array} \right\}$$

For Row, the strategy set is simply $\{C, D\}$.

The normal form is:

| | | **Column** | | | |
|---|---|---|---|---|---|
| | | $(C\|C,\ C\|D)$ | $(C\|C,\ D\|D)$ | $(D\|C,\ D\|D)$ | $(D\|C,\ C\|D)$ |
| | *Deny* | -10,0 | -1, -1 | -1, -1 | -10,0 |
| **Row** | | | | | |
| | *Confess* | -8,-8 | -8, -8 | 0, -10 | 0, -10 |

*Payoffs to: (Row, Column)*

The question did not ask for the Nash equilibrium, but it is disappointing not to know it after all that work, so here it is:

| **Equilibrium** | **Strategies** | **Outcome** |
|---|---|---|
| $E_1$ | {*Confess*, ( *C*\|*C*) (*C*\|*D*)} | Both pick *Confess*. |

115x. (10 points) What is the Nash equilibrium of the following game?

**"A Tax Game"**

| | | **Taxpayer** | |
|---|---|---|---|
| | | *Pay Tax* ($\gamma$) | *Cheat* ($1-\gamma$) |
| | *Audit* ($\theta$) | 2,2 | $3, 0$ |
| **IRS:** | | | |
| | *No Audit* ($1-\theta$) | $3, 3$ | $0, 4$ |

*Payoffs to: (Government, Taxpayer).*

2 Information

2.1x. The boss is trying to decide whether Smith's energy level is high or low. He can only look in on Smith once during the day. He knows if Smith's energy is low, he will be yawning with a 50 percent probability, but if it is high, he will be yawning with a 10 percent probability. Before he looks in on him, the boss thinks that there is an 80 percent probability that Smith's energy is high, but then he sees him yawning. What probability of high energy should the boss now assess?

ANSWER: What we want to find is $Prob(High|Yawn)$. The information is that $Prob(High) = .80$, $Prob(Yawn|High) = .10$, and $Prob(Yawn|Low) = .50$. Using Bayes Rule,

$$Prob(High|Yawn) = \frac{Prob(High)Prob(Yawn|High)}{Prob(High)Prob(Yawn|High) + Prob(Low)Prob(Yawn|Low)} = \frac{(.8)(.1)}{(.8)(.1) + (.2)$$

2.2x. Suppose that Column gets to choose which of the following two payoff structures applies to the simultaneous-move game he plays with Row. Row does not know which of these Column has chosen.

**Payoffs A: "The Prisoner's Dilemma"**

| | | **Column** | |
|---|---|---|---|
| | | *Deny* | *Confess* |
| | *Deny* | -1,-1 | -10, 0 |
| **Row:** | | | |
| | *Confess* | 0,-10 | -8,-8 |

*Payoffs to: (Row,Column).*

**Payoffs B: "A Confession Game"**

| | | **Column** | |
|---|---|---|---|
| | | *Deny* | *Confess* |
| | *Deny* | -4,-4 | -12, -200 |
| **Row:** | | | |
| | *Confess* | -200,-12 | -10,-10 |

*Payoffs to: (Row,Column).*

(a) What is one example of a strategy for each player?

ANSWER. Column: A, Confess. Row: Confess. (I added this question to my first version because most students failed to realize that Column's strategy had two actions and Row's just one)

(b) Find a Nash equilibrium. Is it unique? Explain your reasoning.

ANSWER. Column: A, Confess. Row: Confess. This is unique. Column will want to pick A, because it has higher payoffs for Column for any pair of subsequent actions, and Row's action does not depend on what Column actually chooses — only on what Column is expected to choose.

As happens so often, truly understanding the idea of Nash equilibrium helps a lot. Students commonly answered this question with three proposed equilibria, but did not test for unilateral deviations by Column from choosing Payoffs B to choosing Payoffs A.

Note that if B did observe Column's choice of A or B, contrary to the assumptions here, then another equilibrium would have Column choosing B and Deny, and Row choosing Deny also.

(c) Is there a dominant strategy for Column? Explain why or why not.

ANSWER. Column does have a dominant strategy. Row has two choices Deny or Confess. If Row chooses Deny, Column should respond with (A, Confess), for a payoff of 0 compared to -4, -200, or -1. If Row chooses Confess, Column should respond with (A, Confess), for a payoff of -8 compared to -10, -12, or -10.

(d) Is there a dominant strategy for Row?Explain why or why not.

ANSWER. Row does not have a dominant strategy. Deny is a best response to (B, Deny), but Confess is a best response to (A, Deny).

(e) Does Row's choice of strategy depend on whether Column is rational or not? Explain why or why not.

ANSWER. Row's choice does depend on whether Column is rational. If Column were irrational, Column might, for example, choose (A, Deny), and Row would want to choose Deny also. If Column is rational, Row can depend on Column choosing (A, Confess).

2.3x. *Bank Runs, Coordination, and Asymmetric Information.* A recent article has suggested that during the Chicago bank run of 1932, only banks that actually had negative net worth failed, even though depositors tried to take their money out of all the banks in town. (Charles Calomiris and Joseph Mason (1998), "Contagion and Bank Failures During the Great Depression: THe June 1932 Chicago Banking Panic", *American Economic Review*, December 1997, 87: 863-883.) A bank run occurs when many depositors all try to withdraw their deposits simultaneously, which creates a cash crunch for the bank since banks ordinarily do not keep much cash on hand, and have lent out most of it in business and home loans.

(a) Explain why some people might model bank runs as coordination games.

(b) Why would the prisoner's dilemma be an inappropriate model for bank runs?

(c) Suppose that some banks are owned by embezzlers who each year secretly steal some of the funds deposited in the banks, and that these thefts will all be discovered in 1940. The present year is 1931. Some depositors learn in 1932 which banks are owned by embezzlers and which are not, and the other depositors know who these depositors are. Construct a game to capture this situation and predict what would happen.

(d) How would your model change if the government introduced deposit insurance in 1931, which would pay back all depositors if the bank were unable to do so?

## 3 Continuous and Mixed Strategies

3.1x. Industry output is
(a.) lowest with monopoly, highest with a Cournot equilibrium
@(b.) lowest with monopoly, highest with a Stackelberg equilibrium.
(c.) lowest with a Cournot, highest with a Stackelberg equilibrium.
(d.) lowest with a Stackelberg, highest with a Cournot equilibrium.

3.2x. Three firms producing an identical product face the demand curve $P = 240 - \alpha Q$, and produce at marginal cost $\beta$. Each firm picks its quantity simultaneously. If $\alpha = 1$ and $\beta = 40$, the equilibrium output of the industry is in the interval
(a) $[0, 20]$
(b) $[20, 100]$
(c) $[100, 130]$
@ (d) $[130, 200]$
(e) $[200, \infty]$

3.3x. Is this triopoly game supermodular?
(a) Yes
@(b) No
(c) Only under some values of $\alpha$
(d) Not enough information is provided to answer

3.4x. In this triopoly game, if $\beta$ increases then the industry output
(a) Rises

@(b) Falls
(c) Might either rise or fall
(d) Stays the same

3.5x. If a player uses mixed strategies in equilibrium,
(a.) All players are indifferent among all their strategies.
(b.) That player is indifferent among all his strategies.
@ (c.) That player is indifferent among the strategies he has a positive probability of choosing in equilibrium.
(d.) That player is indifferent among all his strategies except the ones that are weakly dominated.
(e) None of the above.

3.6x. Find the unique Nash equilibrium of the game in Table 3.1.

**Table 3.1: A Game for the 1996 Midterm**

| | | | **Column** | |
|---|---|---|---|---|
| | | *Left* | *Middle* | *Right* |
| | *Up* | 1,0 | 10, −1 | 0, 1 |
| **Row:** | *Sideways* | −1, 0 | -2,-2 | −12, 4 |
| | *Down* | 0, 2 | 823,−1 | 2, 0 |

*Payoffs to: (Row, Column).*

ANSWER. The equilibrium is in mixed strategies. Denote Row's probability of $Up$ by $\gamma$ and Column's probability of $Left$ by $\theta$. Strategies $Sideways$ and $Middle$ are strongly dominated strategies, so we can forget about them. Row has no reason ever to choose $Sideways$, and Column has no reason ever to choose $Middle$.

In equilibrium, Row must be indifferent between $Up$ and $Down$, so

$$\pi_R(Up) = \theta(1) + (1-\theta)(0) = \pi_R(Down) = \theta(0) + (1-\theta)(2)$$

This yields $\theta^* = 2/3$. Column must be indifferent between *Left* and *Right*, so

$$\pi_R(Left) = \gamma(0) + (1-\gamma)(2) = \pi_R(Right) = \gamma(1) + (1-\gamma)(0)$$

This yields $\gamma^* = 2/3$.

3.7x. Three companies provide tires to the Australian market. The total cost curve for a firm making Q tires is TC = 5 + 20Q, and the demand equation is P = 100-N, where N is the total number of tires on the market.

According to the Cournot model, in which the firms's simultaneously choose quantities, what will the total industry output be?

ANSWER. Marginal cost is 20 for each firm. For firm 1, revenue is

$$R_1 = PQ_1 = (100 - Q_1 - Q_2 - Q_3)Q_1,$$

so marginal revenue is $100 - 2Q_1 - Q_2 - Q_3$. Setting this equal to marginal cost yields $20 = 100 - 2Q_1 - Q_2 - Q_3$. Since each firm produces the same quantity in equilibrium, $4Q_1 = 80$, and $Q_1 = 20$. Total industry output is therefore 60.

3.8x (hard). On his job visit, Professor Schaffer of Michigan told me that in a Cournot model with a linear demand curve $P = \alpha - \beta Q$ and constant marginal cost $C_i$ for firm $i$, the equilibrium industry output $Q$ depends on $\Sigma_i C_i$, but not on the individual levels of $C_i$. I may have misremembered. Prove or disprove this assertion. Would your conclusion be altered if we made some other assumption on demand? Discuss.

ANSWER. Everybody had trouble with this. A good approach when stymied is to start with a simple case. Here, the two-firm problem is the obvious simpler case. Prove the proposition for the simple case, and then use that as a pattern to extend it. (Also, you can disprove a general proposition using a simple counterexample, though you cannot prove one using a simple example.)

Note that you cannot assume symmetry of strategies in this game. It is plausible, though not always correct (remember Chicken), when players are identical, but they are not here— firms have different costs. So we would expect their equilibrium outputs to differ.

Also, remember to answer all parts of test questions. The second part of this question asked about nonlinear demand functions, and it is actually the easier part.

The proposition is true.

$$\pi_j = (\alpha - \beta\Sigma_i Q_i - C_j)Q_j,$$

so

$$\frac{d\pi_j}{dQ_j} = \alpha - \beta\Sigma_{i\neq j}Q_i - 2\beta Q_j - C_j = 0,$$

and

$$Q_j = \frac{C_j - \alpha - \beta\Sigma_{i\neq j}Q_i}{2\beta}.$$

Industry output is

$$\Sigma_j Q_j = \Sigma_j \frac{C_j - \alpha - \beta\Sigma_{i\neq j}Q_i}{2\beta} = \Sigma_j \frac{C_j - \alpha}{2\beta} - \Sigma_j \frac{\Sigma_{i\neq j}Q_i}{2}.$$

The first term of this last expression depends on the sum of the firms' cost parameters, but not on their individual levels. The second term adds up the outputs of all but one firm $N$ times, and so equals $(N-1)$ times the sum of the output, $(N-1)\Sigma_j Q_j$. Thus,

$$\Sigma_j Q_j = \Sigma_j \frac{C_j - \alpha}{2\beta N}.$$

This does not depend on the cost parameters except through their sum. Q.E.D.

Chris Pope pointed out one caveat. This proof implicitly assumed that every firm had low enough costs that it would produce positive output. If it produces zero output, it is at a corner solution, and the first order condition does not hold, so the proof fails. Thus, the validity of the proposition depends on the following being true for every $j$:

$$Q_j = \frac{C_j - \alpha - \beta\Sigma_{i\neq j}Q_i}{2\beta} > 0.$$

This condition is not stated in terms of the primitive parameters (it depends on $\Sigma_{i\neq j}Q_i$), so to be quite proper I ought to solve it out further, but I will not do that here.

The result does depend on linear demand. This can be shown by counterexample. Suppose $P = \alpha - \beta Q^2$. Then, attempting the construction above,

$$\pi_j = (\alpha - \beta(\Sigma_i Q_i)^2 - C_j)Q_j,$$

so

$$\frac{d\pi_j}{dQ_j} = \alpha - 3\beta Q_j^2 - 2\beta\Sigma_{i\neq j}Q_iQ_j - C_j = 0.$$

Solving this for $Q_j$ will involve taking a square root of $C_j$. But if $Q_j$ is a function of the square root of $C_j$, then increasing $C_j$ by a given amount and decreasing $C_l$ by the same amount will *not* keep the sum of $Q_j$ and $Q_l$ the same, unlike before, where $Q_j$ was a linear function of $C_j$. So the proposition fails for quadratic demand, and, more generally, whenever demand is nonlinear.

**3.9x: Alba and Rome: Asymmetric information and mixed strategies.** A Roman, Horatius, unwounded, is fighting the three Curiatius brothers from Alba, each of whom is wounded. If Horatius continues fighting, he wins with probability 0.1, and the payoffs are (10,-10) for (Horatius, Curiatii) if he wins, and (-10,10) if he loses. With probability $\alpha = 0.5$, Horatius is panic-stricken and runs away. If he runs and the Curiatii do not chase him, the payoffs are (-20, 10). If he runs and the Curiatius brothers chase and kill him, the payoffs are (-21, 20). If, however, he is not panic-stricken, but he runs anyway and the Curiatii give chase, he is able to kill the fastest brother first and then dispose of the other two, for payoffs of (10,-10). Horatius is, in fact, not panic-stricken.

(a) With what probability $\theta$ would the Curiatii give chase if Horatius were to run?

*Answer.* In a mixed-strategy equilibrium,

$$\pi_h(run) = \pi_h(not\ run), \tag{1}$$

so

$$\theta(10) + (1-\theta)(-20) = 0.1(10) + 0.9(-10) \tag{2}$$

and $\theta^* = \frac{12}{30} = 0.4$.

(b) With what probability $\gamma$ does Horatius run?
*Answer.* In a mixed-strategy equilibrium,

$$\pi_c(chase) = \pi_c(not\ chase), \tag{3}$$

so

$$\begin{array}{c}\alpha(20) + (1-\alpha)\gamma(-10) + (1-\alpha)(1-\gamma)[0.1(-10) + 0.9(10)] = \\ \alpha(10) + (1-\alpha)\gamma(10) + (1-\alpha)(1-\gamma)[0.1(-10) + 0.9(10)],\end{array} \tag{4}$$

which reduces to

$$20\alpha - 10\gamma + 10\alpha\gamma = 10\alpha + 10\gamma - 10\alpha\gamma, \tag{5}$$

so $\gamma^* = \frac{\alpha}{2-2\alpha}$, which equals 0.5 if $\alpha = 0.5$.

(3.9c) How would $\theta$ and $\gamma$ be affected if the Curiatii falsely believed that the probability of Horatius being panic-stricken was 1? What if they believed it was 0.9?
*Answer.* If $\alpha = 1$ (or if the Curiatii think $\alpha = 1$), there is no mixed-strategy equilibrium, because $\pi_c(chase) > \pi_c(not\ chase)$ under all possible circumstances. Thus, the equilibrium is $\gamma^* = 1$, $\theta^* = 1$: Horatius runs and the Curiatii chase. If $\alpha = 0.9$, it is still true that $\pi_c(chase) > \pi_c(not\ chase)$ even if $\gamma = 1$, so the equilibrium is the same as if $\alpha = 1$.

3.10x. (20 points) Find all of the Nash equilibria for the following game.

**Table 1 A Takeover Game**

| | | **Target** | | |
|---|---|---|---|---|
| | | *Hard* | *Medium* | *Soft* |
| | *Hard* | -3, -3 | $-1, 0$ | $4, 0$ |
| **Raider:** | *Medium* | $0, 0$ | 2,2 | $3, 1$ |
| | *Soft* | 0,0 | $2, 4$ | $3, 3$ |

*Payoffs to: (Raider, Target).*

ANSWER. The three equilibria are in pure strategies: (Hard, Soft), (Medium, Medium), and (Soft, Medium).

There are two mixed strategy equilibria.

(1) ( Raider: Hard. Target: Mix between Medium and Soft.) Notice that for Target, Hard is a dominated strategy. That means it will not be part of any mixed strategy equilibrium. Next, notice that Soft is weakly dominated. Thus, if Raider ever plays anything but Hard, Target will want strongly to play Medium. What if Raider plays Hard? Then Target would be willing to mix between Medium and Soft. If Target plays Medium with a probability of $\gamma$, Raider's payoff from Hard is $(-1)\gamma + 4(1-\gamma)$, whereas his payoff from Medium or Soft is $(2)\gamma + 3(1-\gamma)$. Equating these yields $\gamma^* =.25$. If $\gamma$ is no bigger than .25, we have a Nash equilibrium.

(2) ( Raider: Mix between Medium and Soft. Target: Medium. ) How about if Target plays Medium and Raider mixes? Raider would only want to mix between Medium and Soft. But that would generate a Nash equilibrium, for any mixing probability, since Raider gets 2 no matter what, and Target prefers Medium no matter what the mixing probability may be.

3.11x. (40 points) Elena and Mary are the two leading figure skaters in the world. Each must choose during her training what her routine is going to look like. They cannot change their minds later and try to alter any details of their routines. Elena goes first in the Olympics, and Mary goes next. Each has five minutes for her performance. The judges will rate the routines on three dimensions, beauty, how high they jump, and whether they stumble after they jump. A skater who stumbles is sure to lose, and if both Elena and Mary stumble, one of the ten lesser skaters will win, though those ten skaters have no chance otherwise.

Elena and Mary are exactly equal in the beauty of their routines, and both of them know this, but they are not equal in their jumping ability. Whoever jumps higher without stumbling will definitely win. Elena's probability of stumbling is $P(h)$, where h is the height of the jump, and P is increasing smoothly and continuously in $h$. (In calculus terms, let $P'$ and $P''$ both exist, and $P'$ be positive) Mary's probability is $P(h) - .1$— that is, it is 10 percent less for equal heights.

Let us define as h=0 the maximum height that the lesser skaters can achieve, and assume that P(0) = 0.

(a) Show that it cannot be an equilibrium for both Mary and Elena to choose the same value for $h$ (Call them $M$ and $E$).

ANSWER. Suppose they did choose the same value, so $h_a = h_b$. Then Mary could deviate and choose a slightly higher $h$, and she would only slightly increase her probability of stumbling, but would win for sure if neither stumbled.

Notice that this is a simultaneous-move game. The two skaters do not skate simultaneously, but they must make their decisions prior to the Olympics, during training, so they effectively are making simultaneous choices.

This question proved surprisingly difficult. It is a good example of a simple proof. Another good exercise would be to do a careful mathematical answer to this question,using the definition of continuity and laying out the payoff functions.

(b) Show for any pair of values $(M, E)$ that it cannot be an equilibrium for Mary and Elena to choose those values.

ANSWER. Part (a) showed that a pair would not be an equilibrium if $M = E$, so suppose $M \neq E$. Then whoever had the higher value of $h$ would want to reduce her value until it was infinitesimally higher than that of the other skater. But if the values became very close, then whoever had the lower value would want to increase her value to become the highest, for the reasons in part (a). Thus, there is no pure strategy equilibrium.

Both (a) and (b) show the power of the seemingly simple idea of Nash equilibrium. Start with a hypothesized equilibrium, and test for whether any player wants to deviate unilaterally. Do that, and the answer flows out naturally. Start by trying to think what each player ought to do as an optimal strategy and you get tied up in knots.

(c) Describe the optimal strategies to the best of your ability. (Do not get hung up on trying to answer this question; my expectations are not high here.)

ANSWER. The equilibrium must be in mixed strategies. Suppose each skater mixes between all the values of $h$ between 0 and some maximal value $\overline{h}$ which is the same for both skaters. The mixing must be all the way down to zero because if it started from some lower bound $L > 0$, then there is no point in assigning any probability to $L$, since it is sure to lose but increases your chances of stumbling over $h = 0$. The maximal value must be the same for both players because there is no point in having a maximal value greater than the other skater– that just increases the chance of stumbling.

The mixing would be over the interval between the two bounds, because if there were a hole in the support for mixing, the same reasoning as in the last paragraph would imply that there is no reason to put probability on the height which is the upper bound of the hole– you are just as likely to win with a height just infinitesimally greater than the lower bound of the hole.

The mixing probabilties would be different for the two skaters because their skills are different, I conjecture, and Mary would have a greater probability of winning. But my conjecture might be wrong, since the marginal incentives for the two players are the same– the better skater's stumbling probability is different by a constant, not by a multiple.

(d) What is a business analogy? Find some situation in business or economics that could use this same model.

ANSWER. One business analogy is to research in a patent race. Two firms are competing to make a discovery first and patent it. If a company moves too quickly, though, it makes a mistake in its research, and fails to make the discovery at all, because it chases down a blind alley.

For the analogy to be valid, it has to parallel the original model in a number of dimensions. Ask yourself the following questions about your analogy. Are there two players? Is the situation a tournament, where only one can win? What is "stumbling" in the situation? What is "height" in the situation? Are the decisions simultaneous?

It is very important for anyone–undergraduate, MBA, or PhD student – to develop the skill of answering questions like (d).

3.12x. Senator Robert Smith of New Hampshire said of the US policy in Serbia of bombing but promising not to use ground forces, "It's like saying we'll pass on you but we won't run the football." ( *Human Events*, p. 1, April 16, 1999.) Explain what he meant, and why this is a strong criticism of U.S. policy, using the concept of a mixed strategy equilibrium. (Foreign students: in American football, a team can choose to throw the football (to pass it) or to hold it and run with it to move towards the goal.) Construct a numerical example to compare the U.S. expected payoff in (a) a mixed strategy equilibrium in which it ends up not using ground forces, and (b) a pure strategy equilibrium in which the U.S. has committed not to use ground forces.

ANSWER. Senator Smith meant that by declaring our action, we have allowed the Yugoslavs to choose a better response (for them) than if we left them uncertain. Thus, the declaration reduces the expected U.S. payoff. Rather than mixing– which means to be unpredictable– we chose a pure strategy.

Ane example can show this. Suppose that the US has the two alternatives of Air and Ground, and the Yugoslavs have the two alternatives of Air Defense and Ground Defense. Air and Air Defense represent policies of just positioning forces for an air war; Ground and Ground Defense represent policies that also prepare for ground war.

Let the payoffs be as in Table Q.

**Table Q: Kosovo War**

| | | **Yugoslavia** | |
|---|---|---|---|
| | | Air Defense ($\gamma$) | Ground Defense |
| **US** | Air ($\theta$) | 0,0 | 1,-1 |
| | Ground | 2,-5 | -2,-2 |

*Payoffs to: (U.S., Yugoslavia).*

(a) In the mixed strategy equilibrium, Yugoslavia chooses its probability of Air Defense to equate the US payoffs from Air and Ground. Thus,

$$\pi_{US}(Air) = \gamma(0) + (1-\gamma)(1) = 2\gamma + (1-\gamma)(-2) = \pi_{US}(Ground). \quad (6)$$

This reduces to $1-\gamma = 2\gamma - 2 + 2\gamma$, so $3 = 5\gamma$, and $\gamma = 3/5$. The U.S. expected payoff from choosing Air is then $\pi_{US}(Air) = \gamma(0) + (1-\gamma)(1) = 1 - 3/5 = .4$.

(b) If the U.S. instead moves first and chooses Air, Yugoslavia will respond with Air Defense, and the U.S. expected payoff is 0.

Thus, by volunteering to move first, the U.S. reduces its payoff.

3.13x. (a) What is the exact form of every Nash equilibrium of the following game?

ANSWER. This game has a unique Nash equilibrium, in mixed strategies. Let the probability of Aid be $\gamma$ and the probability of Reform be $\theta$. Equating the IMF payoffs yields

$$\pi_{IMF}(Aid) = \theta(3) + (1-\theta)(-1) = -1 * \theta + (1-\theta)(0) = \pi_{IMF}(No\ Aid). \quad (7)$$

This boils down to $3\theta - 1 + \theta = -\theta$, so $5\theta = 1$ and $\theta = .2$, the equilibrium probability of Reform.

Equating the Debtor payoffs yields

$$\pi_{Debtor}(Reform) = \gamma(2) + (1-\gamma)(3) = 1 * \gamma + (1-\gamma)(0) = \pi_{Debtor}(Waste). \quad (8)$$

This boils down to $2\gamma + 3 - 3\gamma = \gamma$, so $6\gamma = 3$ and $\gamma = .5$, the equilibrium probability of Aid.

**Table 5: IMF Aid.**

| | | **Debtor** | |
|---|---|---|---|
| | | Reform | Waste |
| **IMF** | Aid | 3,2 | -1,3 |
| | No Aid | -1,1 | 0,0 |

*Payoffs to: (IMF, Debtor).*

(b) For what story would this matrix be a good model?

ANSWER. The story being modelled here is that the IMF wants to help a debtor country which is trying to reform its economy, but does not want to send aid if it will be wasted by corrupt officials or wasted by giveaways to taxpayers. The debtor country would like to waste the money and avoid reforms, but would accept reforms if that were necessary to get the aid. Both sides must decide in advance which policy they are undertaking– the IMF cannot contract to give the aid only if reforms occur.

## 4 Dynamic Games with Symmetric Information

4.1x. It would seem that all human males must have the same strength of sex drive, because a more motivated male would be more successful in his reproduction, mating with more or better females. In fact, sex drives seem to differ. Use the idea behind the Hawk-Dove model to explain this.

ANSWER. Males with strong sex drives are like Hawks. They more aggressively pursue females, but this means they use up more of their resources in the pursuit, without any corresponding gain if they must compete with other males with strong sex drives. Males with weak sex drives are like Doves. They devote little energy to reproduction, and hence do badly in competition with highly sexed males, but they do fine in competition with each other and can more easily survive.

In equilibrium, both types would persist. If there were too many males with strong sex drives, it would be more advantageous to have a weak sex drive, and waste less energy in fruitless pursuit while winning the occasional female by luck or lack of any competition at all. If there were too many males with weak sex drives, it would be more advantageous to have a strong sex drive, and devote more energy to snapping up the females against the feeble competition.

**4.2x. Dynamic Programming.** (This problem harks back to the War of Attrition in Section 3.2, and forward to the repeated games of Chapter 5.) Firms 1 and 2 are rivals, constantly upgrading their product. One firm is always the leader, the other the follower in technology. The leader earns L per year while the follower earns F. These values L and F are the present values of a year's earnings, so you can think of them as a lump sum paid at the start of the year. A firm can spend either 0 or X on research each year, and the discount rate is r. The leader firm must choose its research level first each year, and then the follower chooses its research level for that year.

Denote the value of the follower firm by $V_f$ and the value of the leader firm by $V_l$, where you must remember that a firm has some probability of changing its state.

The transition matrix, shown below, indicates the probability of the follower becoming the leader under the various combinations of research spending:

| | | FOLLOWER 0 | FOLLOWER X |
|---|---|---|---|
| LEADER | 0 | .1 | .3 |
| LEADER | X | 0 | .2 |

Assume that this is a "Markov" situation: a player's optimizing choice depends on his current state (leader or follower) and not on past history. (This is a dangerous assumption, but I think it holds here for the parameter values we will use.) This means that there are only two decisions to consider: how much research to do in the follower state, and how much to do in the leader state.

(a) If the firm's positions are frozen into place, so that neither firm spends anything on research and the leader and follower never switch places, what are Vf and VL in terms of the exogenous variables? If X=5, L=10, F=1, and r=.10, what are the numerical values? (You can use either dynamic programming or formulas.)

ANSWER. In this case, the firm's values are just the present values of earnings,

$$\begin{aligned} V_F &= F + \sum_1^\infty \left(\frac{1}{1+r}\right) F \\ &= F + \frac{F}{r} = \frac{F(1+r)}{r}. \end{aligned}$$

I took that second step from memory, but you can also use the dynamic programming approach, to derive the perpetuity equation:

$$V_F = F + \left(\frac{1}{1+r}\right) V_F$$

so

$$V_F \left(\frac{1+r-1}{1+r}\right) = F$$

and

$$V_F = \left(\frac{1}{1+r}\right) = \left(\frac{1+r}{r}\right) F.$$

The analogous procedure is followed for the leader to get $V_L = \left(\frac{1+r}{r}\right) L$.

If X=5, L=10, F=1, and r=.10, then $V_F = \left(\frac{1+.1}{.1}\right) 1 = 11$, and $V_L = 110$.

(b) If only the leader spends X on research, what are the values of Vf and VL? If X=5, L=10, F=1, and r=.10, what are the numerical values?

ANSWER. The positions of the two firms will never change. Vf is the same as in part (a). VL is slightly changed, because its net earnings are not L each period, but (L-X), so $V_L = \left(\frac{1+r}{r}\right)(L-X)$. Thus, the values are: $V_F = 11$ and $V_L = 55$.

(c) If neither leader nor follower ever spends anything on research, what are the values of VF and VL in terms of the exogenous variables? If X=5, L=10, F=1, and r=.10, what are the numerical values? (Hint: Find VF first. Then reason by analogy to find VL, without going through all the same algebra again.)

ANSWER.

$$\begin{aligned} V_F &= F + .1\left(\tfrac{1}{1+r}\right) V_L + .9\left(\tfrac{1}{1+r}\right) V_F \\ V_L &= L + .9\left(\tfrac{1}{1+r}\right) V_L + .1\left(\tfrac{1}{1+r}\right) V_F \end{aligned}$$

Getting to the answer is now just solving two equations for two unknowns.

$$V_F \left(\frac{1+r-.9}{1+r}\right) = F + \left(\frac{.1}{1+r}\right) V_L$$

so

$$V_F = \left(\frac{1+r}{.1+r}\right) F + \left(\frac{.1}{.1+r}\right) V_L.$$

Similarly, $V_L \left(\frac{1+r-.9}{1+r}\right) = L + .1 \left(\frac{1}{1+r}\right) V_F$, so $V_L = \left(\frac{1+r}{.1+r}\right) L + \left(\frac{.1}{.1+r}\right) V_F$. Substituting,

$$\begin{aligned} V_F &= \left(\tfrac{1+r}{.1+r}\right) F + \left(\tfrac{.1}{.1+r}\right) \left[\left(\tfrac{1+r}{.1+r}\right) L + \left(\tfrac{.1}{.1+r}\right) V_F\right] \\ &= \left(\tfrac{1+r}{.1+r}\right) F + \left(\tfrac{.1(1+r)}{(.1+r)^2}\right) L + \left(\tfrac{.1^2}{(.1+r)^2}\right) V_F \end{aligned}$$

We can then write

$$V_F \left(\frac{(.1+r)^2 - .01}{(.1+r)^2}\right) = \left(\frac{(1+r}{(.1+r)^2}\right) ((.1+r)F + .1L),$$

so, finally,

$$V_F = \left(\frac{1+r}{r^2+.2r}\right) ((.1+r)F + .1L).$$

The only difference between being the leader and being the follower is that F and L are switched—- the first transition reduces earnings rather than increasing them. Thus,

$$V_L = \left(\frac{1+r}{r^2+.2r}\right) ((.1+r)L + .1F).$$

If X=5, L=10, F=1, and r=.10, then $V_F = \left(\frac{1+.1}{.1^2+.2(.1)}\right) ((.1+.1)(1) + .1(10)) = (111/3)1.2 \approx 45$, and $V_L = \left(\frac{1+.1}{.1^2+.2(.1)}\right) ((.1+.1)(10) + .1(1)) = (111/3)2.1 = 77.7$.

Note that even this very small transition probability of .1 has drastically closed the value gap between the two firms.

I DIDN'T ASSIGN THE FULL QUESTION, BUT YOU MIGHT BE INTERESTED IN WHAT FOLLOWS.

(d) If both firms spend X on research, what are the values of Vf and Vl? If X=5, L=10, F=1, and r=.10, what are the numerical values? (You should reason by analogy from part (c), rather than going through all the algebra again).

ANSWER.

$$\begin{aligned} V_F &= \left(\frac{1+r}{r^2+2(.2)r}\right)((.2+r)(F-X)+.2(L-X)) \\ &= \left(\frac{1.1}{.01+2(.2)(.1)}\right)((.2+.1)(1-5)+.2(10-5)) = (22)(-.2) = -4.4. \end{aligned}$$

$$\begin{aligned} V_L &= \left(\frac{1+r}{r^2+2(.2)r}\right)((.2+r)(L-X)+.2(F-X)) \\ &= \left(\frac{1.1}{.01+2(.2)(.1)}\right)((.2+.1)(10-5)+.2(1-5)) = (22)(.7) = 15.4. \end{aligned}$$

(e) Test for zero research spending being an equilibrium when X=5, L=10, F=1, and r=.10. The equilibrium should be subgame perfect even when non-Markov deviations are allowed, which means that it is not strictly correct simply to compare the payoffs from the different Markov strategies you just calculated. To test a Nash equilibrium, you see if a deviation in a player's own strategy can raise his payoff, keeping the other firm's strategy the same (rather than keeping the *follower strategy* the same). Could either player profitably deviate by spending 5 on research this year, assuming that the other player stays with 0 research now and forever? You do not have to check the equilibrium rigorously, which would require checking all possible deviations, including the optimal responses of the other player, but do speculate on whether zero research is an equilibrium.

ANSWER. The equilibrium is for each firm to spend 0 on research. It simply is not worth changing the transition probability, given the cost. Checking the simplest deviation, if the follower spends 5, he can obtain a .2 increase in the probability that he becomes the leader, and the question is whether his value goes up by more than (1.1)5/.2=27.5. It seems that it is not; being the leader in the zero-research situation is only about 22 better than being the follower.

The leader gets even less benefit from one-time research. At a cost of 5, he avoids a .1 probability of losing his position, so his benefit from research is only half that of the follower's.

**4.3x. The Lions Problem**. (Adapted from a problem told to me by Michael Alexeev, who read something like it in Posner's 5th edition.) One lamb and a number of lions are on a desert island. The island has rats, but the lions would prefer to eat lamb or lion (they are cannibalistic). If a lion eats either a lamb or another lion, however, it becomes sleepy and must take a nap, in which case it is vulnerable to being eaten.

(a) What happens if there are two lions and one lamb?

(b) What happens if there are three lions and one lamb?

(c) What is some situation in business or politics that could be modelled by this story?

## 5 Reputation and Repeated Games with Symmetric Information

5.1x. Suppose that the Battles of the Sexes is repeated an infinite number of times, without discounting. Find a subgame perfect equilibrium strategy profile in which the two players go to different events for the first three repetitions, and thereafter go to the Ballet.

ANSWER: One such strategy profile is: *Man: Ballet, Ballet, Ballet, and thenceforth go to the Ballet unless someone deviated in the first three repetitions, in which case mix between Fight and Ballet in the one-shot mixed strategy equilibrium proportions every time after the deviation occurs. Woman: Fight, Fight, Fight, and thenceforth go to the Ballet unless someone deviated in the first three repetitions, in which case mix between Fight and Ballet in the one-shot mixed strategy equilibrium proportions every time after the deviation occurs.*

If either player deviates in the first three repetitions, the long-term behavior follows the mixed strategy, which has a much lower payoff than BB does for either the Man or the Woman.

5.2x. A company has profits of 10 per year, (paid at the end of the year) but will go bankrupt with probability .2. The interest rate is 5 percent. How much is the company worth?

ANSWER.

$$V = (1/1.05)(.8)(V + 10) = .8(V + 10)/1.05,$$

so

$$\frac{1.05 - .8}{.1.05} V = 10 + V$$

, and $V = 10(.80/.25) = 32$.

Note: I also gave credit for another answer, based on the 10 being paid out even if the firm went bankrupt. Also note that even though the value with 0 percent chance of bankruptcy is 200 (=10/.05), the value with a 20 percent chance is not 160, but only 32. The amount 160 would be the value of a profit stream each year of 10 with probability .8, 0 with probability .2. Bankruptcy, however, ends not only next year's profits, but *all* future years.

5.3x. Each of $N$ "Premium" firms has quality cost $c = \underline{c}$ and each of $D$ "Discount" firms has quality cost $c = \overline{c}$, where $\underline{c} < \overline{c}$. Each firm chooses quality to be $q = 0$, which costs 0 or $q = 1$, which costs $c$. Consumers do not observe quality. Each firm also chooses a one-time level of conspicuous spending at time zero, $S \geq 0$, and a price, $P$. Each of $N$ consumers decides whether to buy, and from which firm, and after buying, the consumer discovers quality for that period. The discount rate per period is $r$. Consumers will pay up to $\overline{P}$ for a good of known high quality and 0 for known low quality, and they maximize consumer surplus. Unless otherwise specified, all payments are made at the end of a period.

(a) If this game is not repeated, so there is only one period, what is the equilibrium? Be sure to specify the complete strategy for each player.

ANSWER: In the last period, $q = 0$ is a dominant strategy for every firm of each type. Consumers will therefore pay no more than 0, and that will be the equilibrium price. The firms have no reason to engage in conspicuous spending, so $S = 0$.

Consumers: Buy if the price is 0; do not buy otherwise. Out of equilibrium beliefs are that firms produce low quality.

Firms: $S = 0$ and low quality. Prices can take any of a variety of levels.

This and part (b) are good examples of the importance of understanding Nash equilibrium and backwards induction. It seems like every test of mine, people lose points for not using these simple concepts.

(b) If this game is repeated three times, what is the equilibrium?

ANSWER: The same as in one period, repeated three times, using backwards induction.

(c) If this game is repeated an infinite number of times, describe an equilibrium in which the $N$ consumers buy from the $N$ Premium firms.

ANSWER: With an infinite number of periods, there are lots of equilibria. Here is one. (7c) is by far the hardest question on the test. I did not expect anyone to get it completely correct, and nobody did.

Consumers: Consumer $i$ starts by buying from Firm $i$ for $i = 1, ...N$. Buy if the price is $p^*$ and the firm spent $S^*$; switch to another firm which has been charging that price and spent $S^*$ otherwise; don't buy if no firm satisfies those conditions. Out-of-equilibrium beliefs: passive conjectures, where needed. [1]

Premium firms: Quality is $q = 1$, spending is $S = S^*$, and price is $p = p^*$. If a firm deviates in any way, it switches to $q = 0$ and $p = 23$ thereafter.

Discount firms: Quality is $q = 0$, spending is $S = 0$, and price is $p = 23$.

To find $S^*$ and $p^*$ we need to do some calculations. A premium firm's continuation payoff from deviating to low quality is $\frac{p^*-0}{1+r}$ for the one period and zero thereafter. Its continuation payoff from high quality is $\frac{p^*-\underline{c}}{r}$. Equating these requires that $rp^* = p^* + rp^* - (1+r)\underline{c}$, so $p^* \geq (1+r)\underline{c}$ will work as far as this kind of deviation goes. Call this Condition (*).

The premium firm's overall equilibrium payoff is $-S^* + \frac{p^*-\underline{c}}{r}$, so we need $S^* \leq \frac{p^*-\underline{c}}{r}$, or the premium firms will deviate to behaving like discount firms, to earn zero payoffs.

[1] Out-of-equilibrium beliefs are not crucial here, since any firm of any type that deviates from equilibrium is expected to produce low quality thereafter.

How about the discount firm? It must spend $S^*$ to fool consumes into thinking it is a premium, for a payoff of $-S^*+p^*$ if it then chooses low quality. $S^* \geq p^*$ will prevent this. Combining the results of these two paragraphs, we need $S^* \in [p^*, \frac{p^*-c}{r}]$. For this range to exist, though, we need $p^* \leq \frac{p^*-c}{r}$, or $p^* \geq \frac{c}{1-r}$. This is more binding than Condition (*); any price which satisfies this condition will satisfy (*) too. So let us take $p^* = \frac{c}{1-r}$ for our equilibrium, and let $S^* = \frac{c}{1-r}$ too.

Another possibility is that the discount firm will deviate by choosing $S = S^*$ and then producing high quality once or forever (if it is profitable for one period, it is profitable forever). Given that $S^* = \frac{c}{1-r}$, however, the premium firms will be making zero profits producing high quality, and the discount firms would make negative profits trying to do so with their higher production costs.

## 6. Dynamic Games with Asymmetric Information

6.1x. Bigfirm Inc., is thinking about taking over the Target Co., which is still controlled by its founder, Mr. Target, even though he does not own much of the stock any more. He is uncomfortable about the takeover, because he loses 120 in utility when he loses his control. With probability .8, Target is in BAD financial shape, and has value 10 for Bigfirm. With probability .2, Target is in GOOD financial shape, and has value 200 for Bigfirm.

Bigfirm only observes the accounting numbers, though. If Target is in BAD shape, the accounting earnings are LOW. If Target is in good shape, Mr. Target gets to choose between HIGH and LOW earnings. If the earnings are HIGH, Bigfirm must pay 100 for the stock; if they are LOW, Bigfirm must pay 60.

Mr. Target receives a utility of 60 from HIGH earnings and 20 from LOW earnings, in addition to whatever else is going on in his utility function.

(a) Draw the game tree for this situation, including the payoffs and information sets.

ANSWER. Natures chooses Bad or Good. If Nature chooses Bad, Target chooses LOW. If Bigfirm then chooses Takeover, Target's payoff is -100 (-120 +20) and Bigfirm's is -50 (10-60). If Bigfirm instead chose Don't Takeover, Target's payoff is 20 (20) and Bigfirm's is 0 (0).

If Nature chooses Good, Target can choose HIGH. If Bigfirm then chooses Takeover, Target's payoff is -60 (-120 +60) and Bigfirm's is 100 (200-100). If Bigfirm instead chose Don't Takeover, Target's payoff is 60 (60) and Bigfirm's is 0 (0). Or Target can choose LOW. If Bigfirm then chooses Takeover, Target's payoff is -100 (-100 +20) and Bigfirm's is 140 (200-60). If Bigfirm instead chose Don't Takeover, Target's payoff is 20 (20) and Bigfirm's is 0 (0).

(b) What is the equilibrium? (hard)

ANSWER. The equilibrium is (LOW, TAKEOVER|HIGH, DON'T|LOW). No out-of-equilibrium beliefs are needed. If Target deviates to HIGH, a takeover will occur, and his payoff falls from 20 to -60. If Bigfirm deviates to DON'T|HIGH, his payoff falls from 100 to 0. If Bigfirm deviates to TAKEOVER|LOW, his payoff falls from 0 to .8(-50) + .2(140) =-40+28 = -12 .

Mistakes to watch out for: (i) Not specifying a strategy for each player, (ii) Forgetting the TAKEOVER|HIGH part of the equilibrium.

(c) What is the equilibrium outcome, in terms of actions taken and expected payoffs from playing this game? (hard)

ANSWER. Nature may choose either GOOD or BAD, Target will always choose LOW, and Bigfirm will always choose DON'T. Target's payoff is 20 and Bigfirm's is 0.

Mistakes to watch out for: (i) Not answering the question (that is, not specifying the actions taken or the expected payoffs).

## 7 Moral Hazard: Hidden Actions

7.1x. In the hidden actions problem facing an employer, inefficiency arises because
@(a.) The worker is risk averse.
(b.) The worker is risk neutral.
(c.) No contract can induce high effort.
(d.) The type of the worker is unknown.
(e.) The level of risk aversion of the worker is unknown.

7.2x. An agent's utility function is U =(log(wage) - effort ). What should his compensation scheme be if different (output,effort) pairs have the probabilities in Table 7.1?
a. The agent should be paid exactly his output.
@b. The same wage should be paid for outputs of 1 and 100.
c. The agent should receive more for an output of 100 than of 1, but should receive still lower pay if output is 2.
d. None of the above.

**Table 7.1: Output Probabilities**

| | | **Output** | | |
|---|---|---|---|---|
| | | 1 | 2 | 100 |
| **Effort** | *High* | 0.5 | 0 | 0.5 |
| | *Low* | 0.1 | 0.8 | 0.1 |

7.3x. For the next few problems, use Table 7.2. The utility function of an agent is $U = w + \sqrt{w} - \alpha e$, and his reservation utility is 0. Principals compete for agents, and have reservation profits of zero. Principals are risk neutral. If $\alpha = 2$, then if the agent's action can be observed by the principal, his equilibrium utility is in the interval
(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
(d) $[10, 40]$
@(e) $[40, \infty]$

**Table 7.2: Output Probabilities**

| | | **Effort** | |
|---|---|---|---|
| | | Low ($e = 0$) | High ($e = 5$) |
| **Output** | $y = 0$ | 0.9 | 0.5 |
| | $y = 100$ | 0.1 | 0.5 |

7.4x. If $\alpha = 10$, then if the agent's action can be observed by the principal, his equilibrium utility is in the interval
(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
@ (d) $[10, 40]$
(e) $[40, \infty]$

7.5x. If $\alpha = 5$, then if the agent's action can be observed by the principal, his equilibrium effort level is
(a) Low
@ (b) High
(c) A mixed strategy effort, sometimes low and sometimes high

7.6x. If $\alpha = 2$, then if the agent's action cannot be observed by the principal, and he must be paid a flat wage, his wage will be in the interval
(a) $[-\infty, 2]$
(b) $[2, 5]$
(c) $[5, 8]$
@ (d) $[8, 12]$
(e) $[12, \infty]$

7.7x. If the agent owns the firm, and $\alpha = 2$, will his utility be higher or lower than in the case where he works for the principal and his action can be observed?
(a) Higher

@ (b) Lower
(c) Exactly the same.

7.8x. If the agent owns the firm, and $\alpha = 2$, his equilibrium utility is in the interval
(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
(d) $[10, 40]$
@ (e) $[40, \infty]$

7.9x. If the agent owns the firm, and $\alpha = 8$, his equilibrium utility is in the interval
(a) $[-\infty, 0.5]$
(b) $[0.5, 5]$
(c) $[5, 10]$
@ (d) $[10, 40]$
(e) $[40, \infty]$

7.10x. A one-man firm with concave utility function $U(X)$ hires a lawyer to sue a customer for breach of contract. The lawyer is risk-neutral and effort averse, with a convex disutility of effort. What can you say about the optimal contract? What would be the practical problem with such a contract, if it were legal?

ANSWER. The contract should give the firm a lump-sum payment and let the lawyer collect whatever he can from the lawsuit. The problem is that the firm would not have any incentive to help win the case.

7.11x. An agent has the utility function $U = log(w) - e$, where $e$ can take the levels 0 and 4, and his reservation utility is $\overline{U} = 4$. His principal is risk-neutral. Denote the agent's wage conditioned on output as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 10. Only the agent observes his effort. Principals compete for agents. Output is as shown in the table below:

| Effort | Probability of Outputs 0 | 10 | Total |
|---|---|---|---|
| *Low* $(e=0)$ | 0.9 | 0.1 | 1 |
| *High* $(e=4)$ | 0.2 | 0.8 | 1 |

What are the incentive compatibility and participation constraints for obtaining high effort?

ANSWER. The incentive compatibility is $.2log(\underline{w}) + .8log(\overline{w}) - 4 \geq .9log(\underline{w}) + .1log(\overline{w}) - 4$.

The participation constraint is $.2log(\underline{w}) + .8log(\overline{w}) - 4 \geq 4$.

Finding these conditions is separate from the issue of whether the principal actually will want to induce high effort.

7.12x. Suppose an agent has the utility function $U = log(w) - e$, where $e$ can take the levels 1 or 3, and a reservation utility of $\overline{U}$. The principal is risk-neutral. Denote the agent's wage conditioned on output as $\underline{w}$ if output is 0 and $\overline{w}$ if output is 100. Only the agent observes his effort. Principals compete for agents, and outputs occur according to the table below.

| Effort | Probability of Outputs 0 | 100 |
|---|---|---|
| *Low* $(e=1)$ | 0.9 | 0.1 |
| *High* $(e=3)$ | 0.5 | 0.5 |

What conditions must the optimal contract satisfy, given that the principal can only observe output, not effort? You do not need to solve out for the optimal contract– just provide the equations which would have to be true. Do not just provide inequalities– if the condition is a binding constraint, state it as an equation.

ANSWER: This is a tricky question because it turns out with these numbers that low effort $(e=1)$ is optimal. In that case, the optimal contract is simple: a flat wage. Because principals compete, a zero-profit constraint must be satisfied, and $w = .9(0) + .1(100) = 10$. The incentive compatibility constraint is an inequality that is not binding: $U(e=1) = log(10) - 1 \geq U(e=3) = log(10) - 3$.

The problem was set up to make it look like high effort was optimal, though, and I did not have you solve out for the entire equilibrium, so I gave full credit for finding the optimal contract for when firms compete to offer high-effort contracts. This is not much more difficult. The contract must satisfy a zero-profit constraint for the principal, and an incentive compatibility constraint for the agent. The zero profit constraint is:

$$.5(0) + .5(100) = .5\underline{w} + .5\overline{w},$$

so $100 = \underline{w} + \overline{w}$.

The incentive compatibility constraint is

$$.5log(\underline{w}) + .5log(\overline{w}) - 3 = .9log(\underline{w}) + .1log(\overline{w}) - 1.$$

That is the constraint, which must be an equality since principals are competing to offer the highest-utility contract to the agent (subject to the zero-profit constraint). Solving out a bit further, $4log(\underline{w}) + 4log(\overline{w}) = 20$, so $log(\underline{w}/\overline{w}) = 5$, and $\underline{w}/\overline{w} = Exp(5) \approx 148$.

The participation constraint for the agent would not be binding.

**7.13x: Bankruptcy Constraints.** A risk-neutral principal hires an agent with utility function $U = w - e$ and reservation utility $\overline{U} = 5$. Effort is either 0 or 10. There is a bankruptcy constraint: $w \geq 0$. Output is given by Table 7.8.

**Table 7.8 Bankruptcy**

| | **Probability of Outputs** | | |
|---|---|---|---|
| **Effort** | 0 | 400 | Total |
| *Low* $(e=0)$ | 0.5 | 0.5 | 1 |
| *High* $(e=10)$ | 0.1 | 0.9 | 1 |

(7.13a) What would be the agent's effort choice and utility if he owned the firm?

*Answer.* $e = 10$, because expected output is then 360 instead of the 200 with low effort, and the agent's utility is 350 instead of 200.

(7.13b) If agents are scarce and principals compete for them what will be the agent's contract under full information? His utility?
*Answer.* Effort is high, as found in part (a). The wage is 360 for high effort and 0 for low (though there are other possibilities). Agent utility is 350.

(7.13c) If principals are scarce and agents compete to work for them, what will the contract be under full information? What will the agent's utility be?
*Answer.* Because principals are scarce, $U = \overline{U} = 5$. Effort is high. The wage is 15 if effort is high, and 0 if it is low.

(7.13d) If principals are scarce and agents compete to work for them, what will the contract be when the principal cannot observe effort? What will the payoffs be for each player?
*Answer.* An efficiency wage must be paid so that the incentive compatibility constraint of part (d) is satisfied. The participation constraint is thus not binding. The low wage will be 0, since the principal wants to make the gap as big as possible between the low wage and the high wage. The high wage must equal 25 to get incentive compatibility. Hence,

$$U = 0.1(0) + 0.9(25) - 10 = 12.5 \tag{9}$$

$\pi(H) = 337.5(= 0.1(0-0) + 0.9(400-25))$. This exceeds $\pi(L) = 195(= 0.5(0-5) + 0.5(400-5))$.

(7.13 e) Suppose there is no bankruptcy constraint. If principals are the scarce factor and agents compete to work for them, what will the contract be when the principal cannot observe effort? What will the payoffs be for principal and agent?
*Answer.* Since agents are risk neutral, selling the store works well. The expected wage must be 15 for the agent so that $U = \overline{U} = 5$, and an incentive compatibility constraint must be satisfied to obtain high effort:

$$0.5w(0) + 0.5w(400) \leq 0.1w(0) + 0.9w(400) - 10, \tag{10}$$

which can be rewritten as $w(400) - w(0) \geq 25$. Many contracts can ensure this. One is to sell the store for 360 minus 10 for the high effort minus 5 for the opportunity cost, which is equivalent to letting the agent keep all the output for a lump-sum payment of 345: $w(0) = 0 + 15 - 360 = -345$ and $w(400) = 400 + 15 - 360 = 55$, which averages to an expected wage of 15 and an expected utility of 5. The principal's payoff is 345.

7.14x. A high-tech firm is trying to develop the game Wizard 1.0. It will have revenues of 200, 000*ifitsucceeds, and*0 if it fails. Success depends on the programmer. If he exerts high effort, the probability of success is .8. If he exerts low effort, it is .6. The programmer requires wages of at least 50, 000*ifhecanexertloweffort, but*70,000 if he must exert high effort. (Let's just use payoffs in thousands of dollars, so 70,000 dollars will be written as 70.)¡p¿

(7.14xa) Prove that high effort is first-best efficient. ¡p¿

ANSWER. You would pay for high effort if you could guarantee it. Profit would then be .8(200) + .2 (0) - 70 = 90. Low effort only yields .6(200) + .4(0) -50 = 70.¡p¿

(7.14xb) Explain why high effort would be inefficient if the probability of success when effort is low were .75. ¡p¿

ANSWER. Profit would then be .8(200) + .2 (0) - 70 = 90 with high effort. Low effort yields .75(200) + .4(0) -50 = 100, which is better. With low effort, the employer can be better off with the worker being no worse off. ¡p¿

(7.14xc) Let the probability of success with low effort go back to .6 for the remainder of the problem. If you cannot monitor the programmer and cannot pay him a wage contingent on success, what should you do? ¡p¿

ANSWER. Pay him 50 to get him to work for you, and expect 70 in profit. ¡p¿

(d) Now suppose you can make the wage contingent on success. Let the wage be S if Wizard is successful, and F if it fails. S and F will have to satisfy two conditions: a participation constraint and an incentive compatibility constraint. What are they? ¡p¿

ANSWER. The participation constraint is that .8S + .2F ¿ = 70. (where I am using ¿ = to mean "greater than or equal to") If this is true, then the programmer will be willing to participate in the development.¡p¿

The incentive compatibility constraint is .8S + .2F ¿ = .6S + .4F + 20, where the 20 represents the extra payoff the programmer would get by slacking off. If this inequality is true, then the programmer has no incentive to slack off, because the payoff of .8S +. 2F that results from high effort is as big as the payoff of .6S + .4F + 20 that results from low effort. ¡p¿

(e) What is a contract that will achieve the first best? ¡p¿

ANSWER.We can rewrite the incentive compatibility constraint as .2 S - .2F ¿ = 20, so S -F ¿ = 100 and S ¿ = 100+F. ¡p¿

Substitute S= 100+F into the participation constraint, and we get .8 (100+F) + .2F ¿ = 70. Making that an equality (so we pay the minimum possible), 80 +.8F + .2F = 70, so F = -10. We will give the programmer a negative wage if the product fails. ¡p¿

We still have to satisfy the participation constraint, though, so we need to pay a generous wage S if the product succeeds. .8S + .2 (-10) = 70, so it must be that .8S = 72, and S =90. ¡p¿

Thus, one of the many contracts that will achieve the first best is ¡b¿ (S =90, F = -10). ¡/b¿ ¡p¿

These two constraints show the big problem in incentive contracts: getting the other side to participate, and getting them to exert the right effort. Much of the structure of wages and salaries can be explained by these two problems. ¡p¿

(f) What is the optimal contract if you cannot pay a programmer a negative wage? ¡p¿

ANSWER. Efficiency wage. Pay him ¡b¿ S=100 and F=0. ¡/b¿Your profit will be 80, compared to 70 with low effort. ¡p¿

7.15x. If a salesman exerts high effort, he will sell a supercomputer this year with probability .9. If he exerts low effort, he will succeed with probability .5. The company will make a profit of 2 million dollars if the sale is made. The salesman would require a wage of \$50,000 if he had to exert low effort, but \$70,000 if he had to exert high effort, he is risk neutral, and his utility is separable in effort and money. (Let's just use payoffs in thousands of dollars, so 70,000 dollars will be written as 70, and 2 million dollars will be 2000)

(a) (5 points) Prove that high effort is first-best efficient.

ANSWER. Suppose the firm could use a forcing contract and choose effort. The expected profit from high effort would be .9(2000) + .1(0) - 70 = 1730. The expected profit from low effort would be .5 (2000) + .5 (0) - 50 = 950. The salesman would have equal payoffs from either effort level. Thus, high effort is Pareto superior.

(b) (5 points) How high would the probability of success with low effort have to be for high effort to be inefficient?

ANSWER. Now let's try equating the two payoffs. They'd be equal if

$$.9(2000) + .1(0) - 70 = 1730 = \theta(2000) + .5(0) - 50, \quad (11)$$

which boils down to $1780 = \theta(2000)$, or $\theta = .89$.

(c) (5 points) If you cannot monitor the programmer and cannot pay him a wage contingent on success, what should you do?

ANSWER. Just pay him a flat wage of 50, since hiring him still yields a profit, but he will choose low effort no matter what you pay him.

(d) (5 points) Now suppose you can make the wage contingent on success. Let the wage be S if he makes a sale and F if he does not. S and F will have to satisfy two conditions: a participation constraint and an incentive compatibility constraint. What are they?

ANSWER. The participation constraint is that .9S + .1F $\geq$ 70.

The incentive compatibility constraint is .9S + .1F $\geq$ .5S + .5F + 20, where the 20 represents the extra payoff the programmer would get by slacking off.

(e) (10 points) What is a contract that will achieve the first best?

ANSWER.We can rewrite the incentive compatibility constraint as .4 S - .4F $\geq$ 20, so S -F $\geq$ 50 and S $\geq$ 50+F.

Substitute S= 50+F into the participation constraint, and we get .9 (50+F) + .1F $\geq$ 70. Making that an equality (so we pay the minimum possible), 45 +.9F + .1F = 70, so F = 35.

We still have to satisfy the participation constraint, though, so we need to pay a generous wage S if the product succeeds. .9S + .1 (35) = 70, so .9S = 73.5, and S =81.7.

Thus, one of the many contracts that will achieve the first best is (S =81.7, F =35).

These two constraints show the big problem in incentive contracts: getting the other side to participate, and getting them to exert the right effort. Much of the structure of wages and salaries can be explained by these two problems.

(f) (5 points) Now suppose the salesman is risk averse, and his utility from money is $log(w)$. Set up the participation and incentive compatibility constraints again.

ANSWER. The participation constraint is

$$.9log(S) + .1log(F) \geq log(70). \tag{12}$$

The incentive compatibility constraint is

$$.9log(S) + .1log(F) \geq .5log(S) + .5log(F) + [log(70) - log(50)], \tag{13}$$

where the term in square brackets represents the extra payoff the programmer would get by slacking off.

(g) (5 points) You do not need to solve for the optimal contract. Using the $log(w)$ utility function assumption, however, will the expected payment by the firm in the optimal contract rise, fall, or stay the same, compared with what it was in part (e) for the risk neutral salesman?

ANSWER. The expected payment will rise. The company will make the payment as small as possible while still meeting the participation constraint. If the salesman were risk neutral, this would be met if the expected wage was $70,000. Since he is risk averse, however, a risk with expected value of $70,000 is not worth enough to meet his participation constraint. The expected value must rise to compensate for the risk.

(h) (5 points) You do not need to solve for the optimal contract. Using the $log(w)$ utility function assumption, however, will the gap between $S$ and $F$ in the optimal contract rise, fall, or stay the same, compared with what it was in part (e) for the risk neutral salesman?

ANSWER. The gap between $S$ and $F$ will fall. Now that the salesman is risk averse, two things are different. First, the bigger the gap, the riskier his wage, and the greater will have to be the expected value so the participation constraint is still satisfied. That means the employer will want to make the gap as small as possible. Second, since a low wage has much lower utility, the punishment of a low wage is more effective, and the incentive compatibility constraint will be satisfied with a smaller gap than if the salesman were risk neutral.

## 8 Topics in Moral Hazard

8.1x. Applying the Revelation Principle to a problem
(a.) Increases the welfare of all the players in the model.
(b.) Increases the welfare of just the player offering the contract.
(c.) Increases the welfare of just the player accepting the contract.
@ (d.) Makes the problem easier to model, but does not raise the welfare of the players.
(e.) Makes the problem easier to model and raises the welfare of some players, but not all.

**8.2x. Mr. Smith is thinking of buying a custom-designed machine from either Mr. Jones or Mr. Brown. This machine costs 5000**

**dollars to build, and it is useless to anyone but Smith. It is common knowledge that with 90 percent probability the machine will be worth 10,000 dollars to Smith at the time of delivery, one year from today, and with 10 percent probability it will only be worth 2,000 dollars. Smith owns assets of 1,000 dollars. At the time of contracting, Jones and Brown believe there is there is a 20 percent chance that Smith is actually acting as an "undisclosed agent" for Anderson, who has assets of 50,000 dollars.**

**Find the price be under the following two legal regimes: (a) An undisclosed principal is not responsible for the debts of his agent; and (b) even an undisclosed principal is responsible for the debts of his agent. Also, explain (as part [c]) which rule a moral hazard model like this would tend to support.**

ANSWER. (a) The zero profit condition, arising from competition between Jones and Brown, is

$$-5000 + .9P + .1(1000) = 0, \tag{14}$$

because Smith will only pay for the machine with probability 0.9, and otherwise will default and only pay up to his wealth, which is 1. This yields $P \approx 5,444$.

(b) If Anderson is responsible for Smith's debts, then Smith will pay the 5,000 dollars. Hence, zero profits require

$$-5000 + .9P + .1(.2)P + .1(.8)(1000) = 0, \tag{15}$$

which yields $P \approx 5,348$.

(c) Moral hazard tends to support rule (b). This is because it reduces bankruptcy and the agent will be more reluctant to order the machine when there is a high chance it is unprofitable. In the model as constructed, this does not arise, because there is only one type of agent, but more generally it would, because there would be a continuum of types of agents, and some who would buy the machine under rule (b) would find it too expensive under rule (a).

Even in the model as it stands, rule (a) leads to the inefficient outcome that a machine worth 2,000 to Smith is not give to Smith. Rather, he pays his wealth and lets the seller keep the machine, which is inefficient since the machine really is worth 2000 to Smith.

*Note:* Nobody answered this question correctly, which surprised me. It basically is a question about zero-profit prices. Guessing would have been a good idea here: it is very intuitive that the price would always be above $5,000, and that it would be higher if the principal never had to cover the agent's debts. You should be able to tell that $P > 10,000$ is impossible, because Smith would never pay it. Also, the sellers compete, so it is their profits that provide a participation constraint, not the benefit to the buyer.

## 9 Adverse Selection

No questions.

## PROBLEMS FOR CHAPTER 9a Mechanism Design in Adverse Selection and in Moral Hazard with Hidden Information

**9a1x. The Groves Mechanism.** A new computer costing 10 million dollars would benefit existing Divisions 1, 2, and 3 of a company with 100 divisions. Each divisional manager knows the benefit to his division (variables $v_i, i = 1, ..., 3$), but nobody else does, including the company CEO. Managers maximize the welfare of their own divisions. What dominant strategy mechanism might the CEO use to induce the managers to tell the truth when they report their valuations? Explain why this mechanism will induce truthful reporting, and denote the reports by $x_i, i = 1, ..., 3$. (You may assume that any budget transfers to and from the divisions in this mechanism are permanent– that the divisions will not get anything back later if the CEO collects more payments than he gives, for example.)

ANSWER: Let Division 1 pay $(10-x_2-x_3)$, Division 2 pay $(10-x_1-x_3)$, and Division 3 pay $(10 - x_1 - x_2)$ if the computer is bought, where that payment could be negative, and buy the computer if $x_1 + x_2 + x_3 \geq 10$.

Manager i's report does not affect its payment except by affecting whether the computer is bought. Let us take the case of Manager 1 for concreteness.

His payoff is $v_1 - (10 - x_2 - x_3)$ if the computer is bought and 0 otherwise. He therefore wants the computer to be bought if and only if $v_1 + x_2 + x_3 \geq 10$. By reporting $x_1 = v_1$, he achieves exactly that outcome– the computer is bought only when he wants it to be bought. If the other two divisions overreport, he wants the computer to be bought because the mechanism will make him pay less than $x_1$, and if they underrport, he wants it not to be bought, because the mechanism will make him pay more than $x_1$.

*9a2x. The Two-Part Tariff.* (Varian 14.10, modified) One way to price discriminate is to charge a lump sum fee $L$ to have the right to purchase a good, and then charge a per-unit charge $p$ for consumption of the good after that. The standard example is an amusement park where the firm charges an entry fee and a charge for the rides inside the park. Such a pricing policy is known as a **two-part tariff**. Suppose that all consumers have identical utility functions given by $u(x)$ and that the cost of production is $cx$. If the monopolist sets a two-part tariff, will it produce more or less than the efficient amount of output?

ANSWER. If the firm produces $x$ units of output and sells it at price $p(x)$ given by the demand curve, then the most it can charge for entry is the consumer surplus, $u(x) - p(x)x$. Once the consumer pays the entry fee, the firm makes a profit of $p(x) - c$ on each unit purchased. The firm's profit maximization problem is thus to maximize by choice of $x$

$$u(x) - p(x)x + (p(x) - cx)x = u(x) - cx.$$

This is solved by setting $u'(x) = c$. This is also the condition for maximizing the sum of producer and consumer surplus, so the monopolist will indeed produce the efficient level of output.

**9a3x: Selling Cars.** A car dealer must pay \$10,000 to the manufacturer for each car he adds to his inventory. He faces three buyers. From the point of view of the dealer, Smith's valuation is uniformly distributed between \$11,000 and \$21,000, Jones's is between \$9,000 and \$11,000, and Brown's is between \$4,000 and \$12,000. The dealer's policy is to make a separate take-it-or-leave-it offer to each customer, and he is smart enough to avoid making different offers to customers who could resell to each other. Use the notation that the maximum valuation is $\overline{V}$ and the range of valuations is $R$.

(9a3xa) What will the offers be?

_*Answer.*_ Let us use units of thousands of dollars. The expected profit from a customer with maximum valuation $\overline{V} > 10$ and range of valuations $R$ is, if price $P$ is charged:

$$\begin{aligned} \pi(P;V,R) &= \int_P^{\overline{V}} \frac{P-10}{R} dV \\ &= \left( \frac{PV}{R} - \frac{10V}{R} \right)\Big|_P^{\overline{V}} \\ &= \frac{\overline{V}P}{R} - \frac{10\overline{V}}{R} - \frac{P^2}{R} + \frac{10P}{R}. \end{aligned} \tag{16}$$

Maximizing profit with respect to $P$ yields the first order condition

$$\frac{d\pi(P;V,R)}{dP} = \frac{\overline{V}}{R} - \frac{2P}{R} + \frac{10}{R} = 0, \tag{17}$$

so

$$P^* = \frac{\overline{V}}{2} + 5. \tag{18}$$

Note that the optimal price does not depend on $R$, the range of possible valuations. Applying (18) to the specific customers: Smith will be offered $P = \frac{21}{2} + 5 = \$15,500$, Jones will be offered $P = \frac{11}{2} + 5 = \$10,500$, and Brown will be offered $P = \frac{12}{2} + 5 = \$11,000$. Moreover, Brown probably values the car less than Jones, but because of the higher probability that he values it more than \$10,000, he will end up paying more if he buys at all.

(9a3xb) Who is most likely to buy a car? How does this compare with the outcome with perfect price discrimination under full information? How does it compare with the outcome when the dealer charges \$10,000 to each customer?

_*Answer.*_ Smith will buy with probability 0.55, which is $\frac{21-15.5}{21-11}$. Jones will buy with probability 0.25. Brown will buy with probability 0.125. Thus, Smith is the buyer most likely to buy.

Whether the dealer charges $10,000 or uses perfect price discrimination, the outcome is the same as far as allocative efficiency: Smith buys with probability 1, Jones buys with probability 0.5, and Brown buys with probability 0.25.

(9a3xc) What happens to the equilibrium prices if with probability 0.25 each buyer has a valuation of $0, but the probability distribution remains otherwise the same?

*Answer.* The prices are the same as in part (a). If a buyer values the car at less than $10,000, it is irrelevant what his value may be, since it is unprofitable to sell to him anyway. Only the part of his distribution above $10,000 matters to the seller's strategy. Note that this has the same flavor as the analysis of auctions, where a bidder's strategy is conditioned on his having the highest valuation, since if he does not, he will generally lose the auction anyway and his bid is irrelevant.

## 10 Signalling

10.1x. If education is to be a good signal of ability,
(a.) Education must be inexpensive for all players.
(b.) Education must be more expensive for the high ability player.
@ (c.) Education must be more expensive for the low ability player.
(d.) Education must be equally expensive for all types of players.
(e.) Education must be costless for some small fraction of players.

10.2x. Suppose that with equal probability a worker's ability is $a_L = 1$ or $a_H = 5$, and that the worker chooses any amount of education $y \in [0, \infty)$. Let $U_{worker} = w - \frac{8y}{a}$ and $\pi_{employer} = a - w$.

There is a continuum of pooling equilibria, with different levels of $y^*$, the amount of education necessary to obtain the high wage. What education levels, $y^*$, and wages, $w(y)$, are paid in the pooling equilibria, and what is a set of out-of-equilibrium beliefs that supports them? What are the self selection constraints?

ANSWER: A pooling equilibrium for any $y^* \in [0, 0.25]$ is

$$w = \left| \begin{array}{l} 1 \; if \; y \neq y^* \\ 3 \; if \; y = y^* \end{array} \right. \tag{19}$$

with the out-of-equilibrium belief that $Pr(L|(y \neq y^*)) = 1$, and with $y = y^*$ for both types.

The self selection constraints say that neither High nor Low workers want to deviate by acquiring other than $y^*$ education. The most tempting deviation is to zero education, so the constraints are:

$$U_L(y^*) = w(y^*) - 8y^* \geq U_L(0) = w(y \neq y^*) \tag{20}$$

and

$$U_H(y^*) = w(y^*) - \frac{8y^*}{5} \geq U_H(0) = w(y \neq y^*). \tag{21}$$

The constraint on the Lows requires that $y^* \leq 0.25$ for a pooling equilibrium.

**10.3x.** Suppose a salesman's ability might be either $x = 1$ (with probability $\theta$) or $x = 4$, and that if he dresses well, his output is greater, so that his total output is $x + 2s$ where $s$ equals 1 if he dresses well and 0 if he dresses badly. The utility of the salesman is $U = w - \frac{8s}{x}$, where $w$ is his wage. Employers compete for salesmen.

(a) Under full information, what will the wage be for a salesman with low ability?

(b) Show the self selection contraints that must be satisfied in a separating equilibrium under incomplete information.

(c) Find all the equilibria for this game if information is incomplete.

ANSWER. (a) Salesmen with low ability would not dress well. Dressing well would raise their output to 3, but their utility at a wage of 3 would be -5, whereas if they dress poorly their utility is 1. Thus, the wage is 1.

(b) In a separating equilibrium, the low-ability salemen must be satisfied with a contract in which they dress poorly, so it must be true that

$$\pi_L(poorly) = w(poorly) \geq \pi_L(well) = w(well) - 8.$$

The high-ability salemen must be satisfied with a contract in which they dress well, so it must be true that

$$\pi_H(poorly) = w(poorly) \leq \pi_H(well) = w(well) - 2.$$

(c) In the separating equilibrium, $w(poorly) = 1$ and $w(well) = 6$. This satisfies the self selection constraints of part (b) and yield zero profits to the employers.

In one pooling equilibrium, $w(poorly) = \theta + 4(1-\theta)$ and $w(well) = 3$ and all salesmen dress poorly, where $\theta$ is the percentage of low-ability salesmen. This is supported by the out-of-equilibrium belief that anyone who dresses well has low ability.

There is no pooling equilibrium in which everyone dresses well. That would require that $w(poorly) = 1$ and $w(well) = \theta + 4(1-\theta) + 2$, and that

$$\pi_L(poorly) = w(poorly) \leq \pi_L(well) = w(well) - 8,$$

so

$$\pi_L(poorly) = 1 \leq \pi_L(well) = \theta + 4(1-\theta) + 2 - 8,$$

but regardless of how close $\theta$ is to 0, this is impossible.

**10.4x.** Explain the difference between an "action" and a "strategy," using a signal jamming game as an example.

ANSWER. An action is a choice a player makes in a game. A strategy is a rule giving the player's choices contingent on each possible information set he might reach in the course of the game.

Consider a signal jamming game of entry deterrence in which the incumbent firm's revenue would ordinarily indicate the size of the market, but in which it can jam that signal by reducing quality and causing revenue to be low even if the market is actually large. The incumbent's *action* is his choice of quality— $Q_1$, for example. His *strategy* is his choice of quality as a function of the size of the market—- $(Q_1|Large, Q_2|Small)$, for example. It may be that the market is almost always small, but the incumbent's strategy must say what quality he will choose in the rare case when the market is large.

*Note:* This was a surprisingly difficult question; only two of eight students did a satisfactory job. It is very important to be able to explain the difference between actions and strategies, and a good exercise would be to do it using five or so different games as examples.

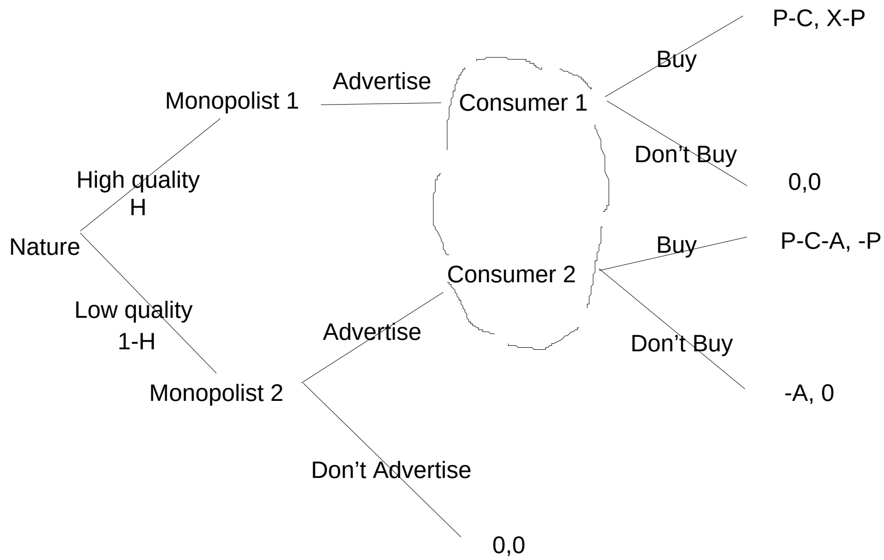
10.5x. A consumer faces a monopoly. He initially believes that the probability that the monopoly has a high-quality product is H, and that a high-quality monopoly would be able to send him an advertisement at zero cost. With probability (1-H), though, the monopoly has low quality, and it would cost the firm A to send an ad. He does receive an ad, offering the product at price P. The consumer's utility from a high-quality product is $X > P$, but from a low quality product it is 0, and the production cost is $C$ for the monopolist regardless of quality, where $C < P - A$.[2]

You may assume that the high-quality firm always sends an ad, that the consumer will not buy unless he receives an ad, and that $P$ is exogenous.

[2]This question did not make it clear whether the cost $C$ was incurred before or after the consumer made his decision, but that does not affect any of the answers except the diagram, where I gave credit either way.

(a) Draw the extensive form for this game.

ANSWER.




(b) What is the equilibrium if H is sufficiently high?

ANSWER. If H is high, then both types of monopoly will advertise, and the consumer will buy the product if he gets an advertisement.

(c) If H is low enough, the equilibrium is in mixed strategies. The high-quality firm always advertises, the low quality firm advertises with probability M, and the consumer buys with probability N. Show using Bayes Rule how the consumer's posterior belief $R$ that the firm is high-quality changes once he receives an ad.

ANSWER. The prior is H. The posterior is

$$R = Prob(High|Advertise) = \frac{Prob(Advertise|High)Prob(High)}{Prob(Advertise)} = \frac{(1)(H)}{(1)(H) + (M)(1-H)}.$$

(d) Explain why the equilibrium is not in pure strategies if H is too low (but $H$ is still positive).

ANSWER. If $H$ is low, then it cannot be an equilibrium for the Low firm always to advertise. Suppose H is close to zero. Then if the Low firm always enters, almost all advertising firms will have low quality, and the consumer will not buy. This would result negative payoffs for the Low firms, so they would not want to advertise.

But neither can it be an equilibrium for no Low firm to advertise. In that case, the consumer would buy, which would make it profitable for the Low firm to advertise.

(e) Find the equilibrium probability of M. (You don't have to figure out N.)

ANSWER. The Low firm's mixing probability M must be such that the consumer is indifferent between buying and not buying. His expected payoff from not buying is 0. From buying, the payoff must be computed using his belief about the probability that the seller has high quality which is the posterior probability $R$. Thus,

$$R(X-P) + (1-R)(-P) = RX - P = \frac{(X)(H)}{(1)(H) + (M)(1-H)} - P$$

Equating this to the payoff of zero from not buying yields $HX = (H + M - MH)P$, so $HX - HP = MP - MHP$ and $M = \frac{H(X-P)}{P(1-H)}$.

**10.6x. Suppose the Federal Reserve Bank is deciding on May 2 whether or not to intervene to support the dollar. With probability** $.4$ **it does not expect the fundamental value of the dollar to be different by July 30. , but with probability .3 it expects the dollar to rise by amount** $X$ **against the yen and with probability .3 it expects the dollar to fall by** $X$**. On May 2, a dollar costs 100 yen. The Fed's objective is to maximize the function** $U = \sqrt{D} + F$**, where** $D$ **is the value of the dollar on May 5 and** $F$ **is the amount of the Fed's profits from trading foreign exchange. On May 2 Fed announces a purchase of** $B$ **dollars to be made on May 3 (where** $B$ **can be a negative number). This trade will yield revenues of** $B$**,** $B(1+X)$ **or** $B(1-X)$**.**

Develop a model of this situation and determine what values $B$ might take.

ANSWER. For the model, use the following order of play.
(0) Nature chooses the dollar to rise to $100 + X$ with probability .3 (good news), fall to $100 - X$ with probability .3 (bad news), and remain unchanged at 100 with probability .4 (no news). The Fed observes Nature's move but the market does not.
(1) The Fed chooses a purchase size $B$, which could be negative.
(2) The market chooses a price $D$ for the dollar which results in zero expected profits.
(3) The Fed purchase is completed.
(4) The true value of the dollar is revealed and the Fed makes profits or losses.

The Fed's payoff function is $U = \sqrt{D} + F$.

Possible strategies include (a) for the Fed to always choose $B = 0$, (b) to choose $B > 0$ if and only if the dollar is going to rise, (c) to choose $B > 0$ always, and (d) to choose $B > 0$ if the dollar is going to rise or if it is going to remain unchanged.

One equilibrium is for the Fed to choose $B = 0$ always, supported by the out-of-equilibrium belief that $B > 0$ implies good news with probability one and $B < 0$ indicates bad news with probability one.

A second equilibrium is for the Fed to buy a very large amount if and only if the news is good. Let us call this amount $B^*$. In equilibrium, the Fed's trading profits will be zero because $D$ will rise to $1 + X$ immediately after it offers to buy $B^*$. If it refrains from buying, $D$ will fall, because goods news is ruled out. In that case,

$$D = D^* = \frac{.3}{.3 + .4}(100 - X) + \frac{.4}{.3 + .4}(100).$$

If there is no news, the Fed's equilibrium payoff is $\sqrt{D^*}$. If it deviates and buys $B^*$, its payoff is

$$\sqrt{100 + X} + B^*(100 - 100 + X).$$

The Fed will not deviate if

$$\sqrt{D^*} \geq \sqrt{100+X} + B^*(100 - 100 + X).$$

This inequality puts a lower bound on $B^*$ as a function of $X$. If $B^*$ is too small, the Fed would deviate to make $D$ rise, taking a loss from trading profits. In this equilibrium $B^*$ can take any value larger than that defined by the last equation.

In a third equilibrium, the Fed buys a limited amount $B^*$ if the news is good or there is no news, and does nothing otherwise. This is supported by the belief that if the Fed buys any other non-negative amount besides 0 or $B^*$, the news is certainly good.

This third equilibrium is fine for the Fed with good news or bad news, but we must check for incentive compatibility for when there is no news. $B^*$ cannot be too large, or when there is no news, the Fed will give up and not trade. If the Fed buys $B^*$, the value of the dollars rises to

$$D = D^* = \frac{.4}{.4+.3}(100) + \frac{.3}{.4+.3}(100+X).$$

The equilibrium payoff for the Fed with no news is

$$\sqrt{D^*} + B^*(100 - D^*).$$

If it deviates and buys zero, its payoff is $\sqrt{100-X}$. Thus, the equilibrium requires

$$\sqrt{D^*} + B^*(100 - D^*) \geq \sqrt{100-X},$$

which means that $B^*$ must not be too large.

Thus, overall, $B$ might take any of a wide range of values in equilibrium, supported by different expectations.

*Note:* Nobody in my class answered this question correctly, and it is genuinely difficult to answer fully. This is a signalling model, and the basic idea is that the Fed will trade off trading profits against a strong dollar. It is also important to realize that when the Fed announces a trade, the market will change the exchange rate, just as the marketmaker changes the asset price in the Kyle model. From there, the first step is to sort out the order of play, and the second step is to figure out an equilibrium. The difficulty arises because there are multiple equilibria. In general, when you are confused, try either (a) writing out an order of play (*without* trying to think about the equilibrium) or (b) telling yourself a story, imagining yourself in the place of the players.

10.7x. A congressional committee has already made up its mind that tobacco should be outlawed, but it holds televised hearings anyway in which experts on both sides present testimony. Explain why these hearings might be a form of signalling, where the audience to be persuaded is congress as a whole, which has not yet made up its mind. You can disregard any effect the hearings might have on public opinion.

ANSWER. This question is taken from work by David Austen-Smith. For the hearings to work as a signal, two things must be true. First, the committee must have some information it wants to convey, and second, it must be most costly to use hearings when the truth is adverse to the committee's desires.

On the first question: the committee wants to convey that outlawing tobacco is desirable. You can think of there being two states of the world (X: Tobacco ban is good) and (Y: Tobacco ban is bad).

On the second question: it must be that holding hearings that make State A seem to be true is less costly if State A is indeed true. The hearings must have credible witnesses, and not be too one-sided, since Congress knows that it is possible, using one-sided hearings, for the committee to equally easily make X or Y seem to be true. The most convincing hearings would be ones with truthful unbiased witnesses, who will say X only if X is true.

Austen-Smith adds another idea for why signalling might work. Hearings are very costly in terms of committee time. If the committee decides to hold hearings, it will not have as much time and budget for other affairs. If the hearing is less likely to support X if X is false, then the willingness of the committee to hold hearings is a good signal in itself, even before any of the evidence is heard.

Still another possibility is that Congress does not know when the committee has made up its mind before the hearing and when it has not. If that is the case, then when the committee is unbiased, it will hold a hearing to help it make up its own mind. The hearing is not a signal in that situation, since an unbiased committee would hold the hearing even if it were secret and even if the rest of Congress did not even know a hearing had been held. Suppose it is not secret, tho. Then a biased committee would also hold a hearing. The motivation of the biased committee is indeed strategic; it wants to make Congress believe the committee is unbiased and has come to its decision after new effort. (This actually would be a somewhat complicated model to flesh out– why does Congress care that the committee has put in effort after the hearing rather than just having an opinion before the hearing?)

This possibility is what is known as a SIGNAL JAMMING model, rather than signalling. It is signal-jamming because it is incomplete information and one type is trying to pretend to be another type by imitating its nonstrategic action, eliminating the informational value in the nonstrategic action.

10.8x. Explain why out-of-equilibrium beliefs must be specified by the modeller in the pooling equilibrium of a signalling game.

ANSWER. In the pooling equilibrium of a signalling game, all the types of players choose the same signal level. If the player being signalled to sees a different signal level chosen, Bayes' Rule cannot tell him what to believe about the type of player who has signalled that way. Thus, the modeller, as part of the equilibrium, must specify his beliefs.

## 11 Bargaining

(11.1x) Smith makes a take-it-or-leave-it offer to Jones of X for his car, which is worth 2000 dollars to Jones and 8000 dollars to Smith. What is X?

ANSWER. 2000 dollars. Jones will accept, in the only Nash equilibrium of this game. 2001 is not a Nash equilibrium offer, because 2000.5 dominates it.

**11.2x. (See Rasmusen negotiation paper.)** Two parties, the Offeror and the Acceptor, are trying to agree to the clauses in a contract. They have already agreed to a basic contract, splitting a surplus 50-50, for a surplus of $Z$ for each player. The offeror can at cost $C$ offer an additional clause which the acceptor can accept outright, inspect carefully (at cost $M$), or reject outright. The additional clause is either "genuine," yielding the Offeror $X_g$ and the Acceptor $Y_g$ if accepted, or "misleading," yielding the Offeror $X_m$ (where $X_m > X_g > 0$) and the Acceptor $-Y_m < 0$.

What will happen in equilibrium?

ANSWER. One equilibrium is for the Offeror never to offer the additional clause and for the Acceptor to believe, out of equilibrium, than any clause offered is Misleading and hence to reject it.

A more interesting equilibrium is in mixed strategies. There is not a pure strategy equilibrium in which the Misleading clause is always offered, because it would always be rejected then. There is not a pure strategy equilibrium in which the Genuine clause is always offered, because there would never be any inspection. This is an auditing game. If the inspection cost, $M$, is not too high, there is a mixed strategy equilibrium.

Let the probability of offering a Genuine clause be $\theta$ and the probability of inspecting be $\gamma$. Equating the payoffs to the Offeror from offering Genuine or Misleading clauses gives us

$$\pi(genuine) = -C + X_g = \pi(misleading) = -C + (1-\gamma)X_m,$$

which solves to $\gamma = 1 - \frac{X_g}{X_m}$.

Equating the payoffs to the Acceptor from Accepting and Inspecting gives us

$$\pi(Accept) = \theta Y_g - (1-\theta)Y_m = \pi(Inspect) = -M + \theta Y_g,$$

which solves to $\theta = 1 - \frac{M}{Y_m}$.

*Note:* This was a moderately difficult question. The key here is to tell yourself a story about what will happen, taking the point of view of each player in turn.

11.3x. A seller with marginal cost constant at $c$ faces a continuum of consumers represented by the linear demand curve $Q^d = a - bP$, where $a > c$. Demand is at a rate of one or zero units per consumer, so if all consumers between points 1 and 2.5 on the consumer continuum make purchases at a price of 13, we say that a total of 1.5 units are sold at a price of 13 each.

(a) What is the seller's profit if he chooses one take-it-or-leave-it price?

ANSWER. This is the simple monopoly pricing problem. Profit is

$$\pi = Q(P - C) = Q(a/b - Q/b - c).$$

Differentiating with respect to Q yields

$$\frac{d\pi}{dQ} = a/b - 2Q/b - c = 0,$$

which can be solved to give us

$$Q_m = \frac{a - bc}{2}.$$

The price is then, using the demand curve,

$$P_m = \frac{a/b + c}{2},$$

which is to say that the price will be halfway between marginal cost and price which drives demand to zero. Profit is

$$\pi_m = \left(\frac{a/b - c}{2}\right)\left(\frac{a - cb}{2}\right).$$

(b) What is the seller's profit if he chooses a continuum of take-it-or-leave-it prices at which to sell, one price for each consumer? (You should think here of a pricing function, since each consumer is infinitesimal).

ANSWER. Under perfect price discrimination, the seller captures the entire area under the demand curve and over the marginal cost curve, because he charges each consumer exactly the reservation price. Since the price at which quantity demanded falls to zero is $a/b$ and the quantity when price equals marginal cost is $a - bc$, the area of this profit triangle is

$$\pi_{ppd} = (1/2)(a/b - c)(a - bc)$$

Note that this is exactly twice the monopoly profit found earlier.

(c) What is the seller's profit if he bargains separately with each consumer, resulting in a continuum of prices? You may assume that bargaining costs are zero and that buyer and seller have equal bargaining power.

ANSWER. In this case, which I call "isoperfect price discrimination," profits are exactly half of what they are under perfect price discrimination, since the price charged to a consumer will exactly split the surplus he would have if the price equalled marginal cost. Thus, the profit is the same as using the simple monopoly price.

## 12. Auctions

12.1x. If I am bidding for a rare coin in a second-price sealed-bid auction, and the coin would be worth 1000 dollars to me, the most reasonable bid below is
(a.) 1100 dollars.
@(b.) 1000 dollars.
(c.) 950 dollars.
(d.) 100 dollars.
(e.) 0 dollars.

## 13. Pricing

13.1x. Two firms submit bids to supply a goverment contract. Firm 1 has known cost $c$. Firm 2 has cost of either $c$, or, with probability $\theta$, infinity. $\theta$ is a probability lying between 0 and 1, inclusive of both end points. The government will buy one unit at the lowest price, paying up to reservation price $R$.

What happens? Be as precise as possible about the prices the two firms charge. (I do not expect most people to analyze this completely correctly. Do the best you can, describing what happens verbally and mathematically. )

ANSWER. If theta is big enough, firm 2 charges R.

If $\theta$ is zero, both firms charge $c$.

If $\theta$ is low, both firms play mixed strategies.

13.2x. A seller faces a large number of buyers whose market demand is given by $P = \alpha - \beta Q$. Production marginal cost is constant at $c$.

(a) What is the monopoly price and profit?

ANSWER: Profit is $PQ - cQ$ or $(\alpha - \beta Q - c)Q$. The first order condition is $\alpha - 2\beta Q - c = 0$, so $Q = \frac{\alpha - c}{2\beta}$. The price is then $P = \alpha - \beta \frac{\alpha - c}{2\beta} = \alpha - \frac{\alpha - c}{2} = \frac{\alpha + c}{2}$. The profit is $(P - c)Q = (\frac{\alpha + c}{2} - c)\frac{\alpha - c}{2\beta} = \frac{(\alpha - c)^2}{4\beta}$.

I was shocked at how badly people did on this undergraduate-level question, and the similarly easy (b).

(b) What are the prices under perfect price discrimination if the seller can make take-it-or-leave-it offers? What is the profit?

ANSWER: Under perfect price discrimination, there is a continuum of prices along the demand curve from $\alpha$ to $c$. The profit equals the area of the triangle under the demand curve and above the flat MC curve, which is $(1/2)(\alpha - c)Q(c) = (1/2)(\alpha - c)\frac{\alpha - c}{\beta} = \frac{(\alpha - c)^2}{2\beta}$. Notice how profit has doubled compared to the simple monopoly profit.

(c) What are the prices under perfect price discrimination if the buyer and sellers bargain over the price and split the surplus evenly? What is the profit?

ANSWER: If buyers and sellers split the surplus evenly, then instead of the seller getting the entire surplus, he only gets half, so profits are half those found in part (b). There is a continuum of prices between $c + \frac{\alpha - c}{2}$ and *c*. The profit is $\frac{(\alpha-c)^2}{4\beta}$, the same as the monopoly profit in this special case.

13.3x. Renting helps the durable monopolist because
(a) it permits him to produce a less durable product.
(b) it rescues him from a Prisoner's Dilemma.
(c) it reduces adverse selection.
@ (d) he is then not tempted to lower his future price.

# 14. Entry

14.1x. Mr. Turner is thinking of entering the garbage collection business in a certain large city. Currently, Cutright Enterprises has a monopoly, earning 40 million dollars from the 40 routes the city offers up for bids. Turner thinks he can take away as many routes as he wants from Cutright, at a profit of 1.5 million per route for him. He is worried, however, that Cutright might resort to assassination, killing him to regain their lost routes. He would be willing to be assassinated for profit of 80 million dollars, and assassination would cost Cutright 6 million dollars in expected legal costs and possible prison sentences.

14.1x. How many routes should Turner try to take away from Cutright?

ANSWER. Turner should take away 5 or 6 routes. Only when he exceeds 6 is it profitable for Cutright to kill him.

Turner's strategy is the number N of routes he takes away. Cutright's strategy is either Kill or Not Kill, as a function of Turner's action.

One perfect equilibrium is: (Turner chooses N=5, Cutright kills if and only if N is 6 or more.) If Turner deviates to N=6, he is killed, reducing his payoff by 80. Cutright is willing to do this because he would gain 6 million from the regained routes at a cost of 6 million. Deviating to killing for N=5 would result in a lower payoff.

A second perfect equilibrium is: (Turner chooses N=6, Cutright kills if and only if N is 7 or more.) Cutright is willing not to kill if N=6 because he is indifferent; killing leaves his payoff unaffected.

Note that Cutright prefers the first equilibrium, but he cannot cause that equilibrium to be the one played out unless he can affect the expectations of both players. Both of these are perfect equilibria.

The WSJ had an op-ed piece by someone like Turner on Dec. 16, 199x. He is trying to break into the New York garbage market, and found that cooperation with the federal prosecutors was the most important part, given Mafia control of business garbage collection. Shortly after he entered, someone left him a dog's head and a message "Welcome to New York." Have the Gambinos read Fudenberg and Tirole?

14.1xb. Explain which of these strategies Turner's optimal strategy most resembles: Fat cat, lean-and-hungry-look, puppy dog, top dog.

ANSWER. This is a puppy dog strategy. Turner is playing a "soft" strategy of limited entry, so that Cutright will play "soft" also. . Turner is refraining from committing to entry on a large scale because he does not want to provoke retaliation from his rival. Commitment would make him tougher, which would make his rival tougher too, so he holds back.

You could perhaps interpret this as a Top Dog strategy of Turner playing Hard, because he chooses N=5 rather than N=0, and Cutright backing off and playing Soft in response. It depends on whether you think entry in 5 routes is an aggressive strategy. I myself would say it is not, and that the interesting choice is between 5 and 40 routes.

14.2x. Two marketing executives are arguing. Smith says that reducing our use of coupons will make us a less aggressive competitor, and that will hurt our sales. Jones says that reducing our use of coupons will make us a less aggressive competitor, but that will end up helping our sales.

Discuss, using the effect of reduced coupon use on your firm's reaction curve, under what circumstance each executive could be correct.

ANSWER. There are a couple of ways to look at this problem.

(1) One way is that the important strategy is coupon use directly. Smith thinks that coupons are strategic substitutes, so when we reduce our use of coupons, our rival will increase their use, and we will be hurt. Jones thinks that coupons are strategic complements, so when we reduce our use of coupons, our rival will reduce their use too, to the benefit of both of us.

(2) A second way is in terms of how coupon use affects how the two companies play a game in the consumer market.

Smith thinks that our firm is in a market with downward sloping reaction curves in the important strategy– strategic substitutes, as with Cournot competition. If we use fewer coupons, that will shift in our reaction curve, and we will end up with lower sales. We need to be "lean and hungry", because if we use coupons to make us softer in the product market, our rival will react by being tougher.

The important strategy might be, for example, output, and if we use more coupons, that will make us less willing to produce high output in reaction to what our rival does, because each sale will be profitable. In the end, we will contract our output and our rival will increase his.

Jones thinks that our firm is in a market with upward sloping reaction curves in the important strategy– strategic complements, as with Bertrand competition. If the important variable is price, and we use fewer coupons, that will shift out our reaction curve, and we will increase our price. So will our rival, and we will both end up with higher profits. We thus adopt a "fat cat" strategy– we use more coupons to make us softer in the product market, and our rival becomes softer in response.

16. Other Topics (Externalities, Miscellaneous, Opportunity Cost)

16.1. A monopolist faces a linear demand curve $q = a - bp$ and has constant marginal cost $c$.

(i) Show that the magnitude of the elasticity of demand is increasing in $b$.

(ii) Compute the welfare loss from monopoly pricing. How does the ratio of the deadweight loss to total welfare vary with $b$?

(iii) Suppose you were told outright that the deadweight loss was monotonic in the slope of the demand curve. Prove without algebra that it must be monotonically *decreasing* in the slope.

ANSWER. (i) The elasticity of demand is $\epsilon = -\frac{p}{q}\frac{dq}{dp}$. With linear demand, $\epsilon = \frac{pb}{a-pb}$. Differentiating with respect to $b$ yields

$$\frac{d\epsilon}{db} = \frac{p}{a-bp} + \frac{bp^2}{(a-bp)^2},$$

which is positive. Thus, the elasticity increases in $b$.

(ii) Keeping the other parameters fixed, let's change $b$. The deadweight loss is the triangle,

$$D(b) = \frac{1}{2}[p_m(b) - c][a - bc - q_m(b)]$$

Differentiate to get

$$D'(b) = \frac{1}{2}\{[p'_m(b)[a - bc - q_m(b)] - [p_m(b) - c][c + q'_m(b)]\},$$

or

$$D'(b) = \frac{1}{2}[p_m(b) - c][p'_m(b)b - c - q'_m(b)].$$

The first two terms of this are positive, because price exceeds cost. To find the second term, we need the monopoly price and output. These are $q_m = (a-bc)/2$ and $p_m = (a+bc)/2b$. Thus, the second term is $-(a+bc)/2b$, which is negative and we can conclude that $D'(b) < 0$. The deadweight loss falls as the demand curve gets steeper.

(iii) The answer, and the intuition behind part (ii), is that as the the demand curve gets steeper, but the price-intercept stays the same, the market is shrinking. If it shrinks enough, the deadweight loss goes to zero, because the entire social surplus goes to zero.

16.2. Why does it not create inefficiency if I bid up the price of the Onassis diamond ring, hurting the other bidders in an auction, while it does create inefficiency if I smoke during the auction, hurting the other bidders?

ANSWER. Pecuniary vs. real externalities.

16.3. The idea behind Chapter 11 of the bankruptcy code is that some firms can avoid bankruptcy if given some temporary relief from creditors. They go to a judge, and persuade him that they have a good chance of ultimately repaying their creditors if they are allowed to delay repayment. It is argued this is socially useful, because it results in fewer firms going bankrupt and less waste of resources. Why does the concept of opportunity cost suggest that this reasoning is wrong, and that immediate bankruptcy is actually a good thing?

ANSWER. Keeping the firm alive has an opportunity cost—- its assets are tied up in that firm, so no other firm can use them. If it goes bankrupt, those resources are freed up.

16.4. A millionaire lives next to a small woods owned by a lumber company. The lumber company will make a good profit if it cuts down all the trees this year, but this would ruin the value of the millionaire's residence.

(a) Explain why it does not make any difference to the survival of the trees whether the law allows the lumber company to cut down the trees without the millionaire's permission or whether it requires the company to get the millionaire's permission first.

(b) What would happen if the law currently did not require permission, but lobbying efforts could change the law in time to be apply to this tree harvest?

ANSWER. Not recorded.

16.5. Explain, using an Edgeworth Box, how, if two agents have convex preferences, and two equilibrium prices are possible starting from a given endowment, it is necessary that the two agents prefer different equilibria.

ANSWER. Not recorded.

16.6. If I say that midterms reduce Craig's utility twice as much as they reduce Chris's, what assumptions am I making about interpersonal comparison, ordinality, and cardinality of utility functions?

ANSWER. The assumption is that utility can be compared interpersonally (Craig vs. Chris), that it is ordinal ( we can say that utility falls because of a midterm), and that it is cardinal (it not only falls, but by an amount which can meaningfully be called "twice" that of someone else).

16.7. Construct a numerical example to show that a monopoly might produce a quality level greater than the social optimum.

ANSWER: Here is one example— the one I used in class. Let there be two consumers, each of whom buys up to one unit. The eager consumer will pay up to 102 for high quality or low quality. The reluctant consumer will pay up to 82 for low quality and 102 for high quality. Low quality costs 2 per unit to produce, while high quality costs 20. The seller cannot observe consumer type.

If he offers just low quality, he should charge a price of 82, for profit of 2(82-2) = 160 and social surplus of (102+82-2(2)) = 180. If he offers just high quality, he should charge a price of 102, for profit of 2(102-20) = 164 and social surplus of (102+102-2(20)) = 164. He can't do better by offering both qualities. Thus, he will offer high quality, but that is not socially optimal.

One thing to watch out for in constructing a two-customer example is whether the seller will choose to sell to both customers, or only one.

16.8. A comedian chooses the offensiveness level of his jokes. His payoff from offensiveness $x$ is $6x - x^2$. His audience gets a payoff of zero because he is able to perfectly price discriminate. Certain other people are offended, and incur cost $2x$.

(a) What is the socially optimal level of offensiveness?

ANSWER: Total surplus is $6x - x^2 - 2x$, or $4x - x^2$. Thus, the first order condition is $4 - 2x = 0$, and $x^* = 2$.

(b) What is the laissez faire level of offensiveness?

ANSWER. 3. That maximizes $6x - x^2$, which has first order condition $6 - 3x = 0$.

(c) Explain in words why the level of offensiveness you found in part (a) is socially optimal.

ANSWER: If the offensiveness level were any higher, the extra benefit to the comedian would be less than the extra cost to the people who are offended. If the offensiveness level were any lower, it should be increased, because then the the extra benefit to the comedian would be greater than the extra cost to the people who are offended.

This proved to be a hard question. Its purpose was to illustrate that not just math, but words can be made precise. A number of bad answers are: "2 solves the social optimization problem," "2 both maximizes the comedian's payoff and minimizes the cost to other people," "2 gives weight to both the comedian's payoff and the cost to others," and "2 maximizes the social surplus." If you got this wrong, go back and read Rhoads again.

4. (15 points) *Externalities.* A factory produces $x$ tons of steel, and, in the process, $x$ pounds of soot. Its profit function is $\pi_1 = px - c(x)$, where $p = 10$ and $c(x) = 2x$ up to a capacity of 10, and infinite thereafter. The neighboring laundry has a payoff function of $\pi_2 = -e(x)$, where $e(x) = x^2$.

(a) What is the socially optimal level of soot, $x^*$? What is the total social surplus?

ANSWER. Social welfare is $px - c(x) - e(x) = 10x - 2x - x^2$. The first order condition is $8 - 2x = 0$, so $x^* = 4$. The surplus is 40-8 -16 = 16.

(b) What is the laissez faire level, $x'$, of soot emission if negotiation between factory and laundry is prohibitively costly? What are the payoffs of each party?

ANSWER. The factory's profit is $\pi_1 = px - c(x) = 10x - 2x$. Maximizing this leads to a corner solution of $x' = 10$. The factory's payoff is 80 and the laundry's is $-100$.

(c) What is the laissez faire level, $x''$, of soot emission if negotiation between factory and laundry is costless? What is the social surplus? What is the payoff of each player?

ANSWER. The two parties will agree on $x'' = 4$, and social surplus will be 16. The factory's profit will be between 80 and 116, depending on bargaining power, and the laundry's payoff will be between $-100$ and $-64$.

(d) What is the laissez faire level, $x'''$, of soot emission if emission is completely banned unless permission from the laundry is obtained, but negotiation between factory and laundry is costless? What is the social surplus? What is the payoff of each player?

ANSWER. The two parties will agree on $x'' = 4$, and social surplus will be 16. The factory's profit will be between 0 and 16, depending on bargaining power, and the laundry's payoff will be between 0 and 16.

6. (10 points) An owner of a tract of land values his land at $v_s$ and a potential buyer values it at $v_b$. The buyer and seller do not know each other's valuations, but guess that they are uniformly distributed between 0 and 1. The seller and buyer suggest $p_s$ and $p_b$ simultaneously, and they have agreed that the land will be sold to the buyer at price $p = \frac{(p_b+p_s)}{2}$ if $p_s \geq p_b$.

The actual valuations are $v_s = .2$ and $v_b = .8$. What is one equilibrium outcome given these valuations and this bargaining procedure? Explain why this can happen.

UNCLASSIFIED AS YET

Emptry right now.