MEASURING RISK

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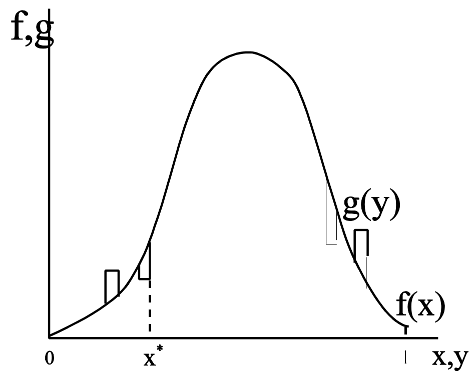
We have defined the meaning of "one individual is more or less risk averse than another", which describe the subjective preference. Now we want to determine under what condition we can say "one gamble is more risky than another", which describe the objective events.

I. Definition

Two risky events X and Y, Y is said riskier than X if :

(1) every risk averse agent prefers X to Y, i.e. $EU(X) \geq EU(Y)$ for any concave U( );
(2) Y is equal to X plus "noise", i.e., $Y = X+e$, with $E(e \mid X) = 0$; or
(3) Y has more weight on the tails of its density function than X, but $E(X)=E(Y)$.



Economists used to define as:

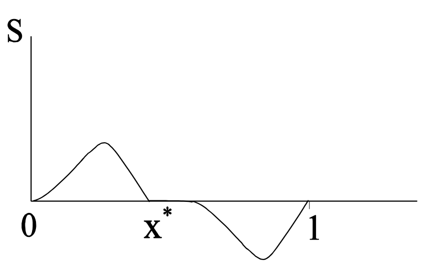
(4) $E(X)=E(Y)$ but $V(X) < V(Y)$. This one is not consistent with others.

II. Proof of the Equivalence of the Three Definitions

Let G be the cumulated density function of Y, and $g(y) = G'(y)$ is the density function. Similarly, let F be the c.d.f. of X, and $f(x) = F'(x)$. We only consider the case when both events have outcomes only between 0 and 1 (others can be scaled).

$S = G - F$ for $x,y \in [0,1]$
$s = g - f = S'$

$S(0) = G(0) - F(0) = 0$
$S(1) = G(1) - F(1) = 0$



As x,y rises above 0 at some point $S > 0$ up to a point x^*.
$S(x^*) = 0$, $G(x^*) = F(x^*)$.
As x,y rises above x^* at some point $S < 0$ and finally $S = 0$.

Mean Preserving Spread:

G(y) is said to be obtained by taking a mean preserving spread from F(x) if

1) S(0) = S(1) = 0;
2) there exists an $x^* > 0$ such that $S(x)\ \$\ 0$ when $x\ \#\ x^*$ and $S(x)\ \#\ 0$ when $x\ \$\ x^*$;
3) $\int_0^1 S(z)dz = 0;$ and
4) $\int_0^m S(z)dz \geq 0,$ œ m 0[0,1).

The third definition in **I** means G(y) is a mean preserving spread from F(x). Check using the rule of integration by parts:

$$
\begin{aligned}
\int_0^1 S(z)dz &= S(z)z\Big|_0^1 - \int_0^1 z dS(z) \\
&= S(1) - S(0)0 - \int_0^1 zs(z)dz \\
&= 0 - \int_0^1 zg(z)dz + \int_0^1 zf(z)dz \\
&= E(x) - E(y) \\
&= 0
\end{aligned}
$$

<u>Sketch Proof</u>: **<u>Step 1</u>**

(2) Y (1)

y = x + e E(e*x) = 0

For any risk averse agent, U( ) is concave.

$E_e U(x + e)\ \#\ U[E_e(x + e)] = U[x + E(e*x)] = U(x)$ by Jensen's Inequality

$E_x E_e\ U(x + e)\ \#\ E\ U(x)$

$E\ U(y)\ \#\ E\ U(x)$

<u>Step 2</u>

(1) Y (3)

For any concave U( ), EU(y) # EU(x)

Denote S = G - F, where G, F are the c.d.f. of y and x, respectively.

S(0) = G(0) - F(0) = 0 - 0 = 0

S(1) = G(1) - F(1) = 1 - 1 = 0

$$\begin{aligned} EU(y) - EU(x) &= \int_0^1 U(z)g(z)dz - \int_0^1 U(z)f(z)dz \\ &= \int_0^1 U(z)\left[g(z) - f(z)\right] \\ &= \int_0^1 U(z)s(z)dz \leq 0 \end{aligned}$$

where g, f and s are the differencial function of G, F and S. Because U( ) can be any concave function, not necessarily strictly concave, we can pick U(z) = z and U(z) = -z. Both I(z)s(z)dz # 0 and I(z)s(z)dz $0 hold. Then, Iz s(z) dz = 0 is the only way to satisfy.

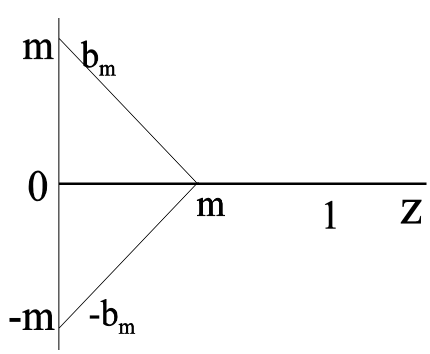
Use the rule of integration by parts:

$$\begin{aligned} \int_0^1 zs(z)dz &= zS(z)\Big|_0^1 - \int_0^1 S(z)dz \\ &= S(1) - \int_0^1 S(z)dz \\ &= 0 \end{aligned}$$

Because S(1) = G(1) - F(1) = 1 - 1 = 0

$$\hat{}\ \int_0^1 \mathrm{S}(\mathrm{z})\mathrm{dz} = 0$$

For m0[0,1), define by $b_m(z) = \max(m-z,0)$, $b_m(z)$ is convex and $-b_m(z)$ is concave. Again we have



$$\int_0^1 -b_m(z)s(z)dz \le 0$$

$$\int_0^1 b_m(z)s(z)dz \ge 0$$

$$\int_0^1 \max(m - z, 0)s(z)dz$$

$$= \int_0^m (m - z)s(z)dz + \int_m^1 0 \cdot s(z)dz$$

$$= \int_0^m (m - z)s(z)dz$$

$$= m\int_0^m s(z)dz - \int_0^m zs(z)dz$$

$$= mS(z)\Big|_0^m - \left[zS(z)\Big|_0^m - \int_0^m S(z)dz \right]$$

$$= mS(m) - 0 - mS(m) + 0 + \int_0^m S(z)dz$$

$$= \int_0^m S(z)dz \ge 0$$

Step 3

(3) Y (2)

Define e = y - x, follow step 2, S and s are the c.d.f. and p.d.f. of e.

E(e*x) = I z s(z)dz = 0 has been proved in Step 2.

Notice, that definition (4) is not equivalent with the first three. Actually, (4) is not a sounding definition, because there are cases that for two risky assets having same means and same variances, one is consistently preferred to the other by all risk averters due to the properties of higher moments, such as skewness and kurtosis.

III. Some Commonly Used Distributions

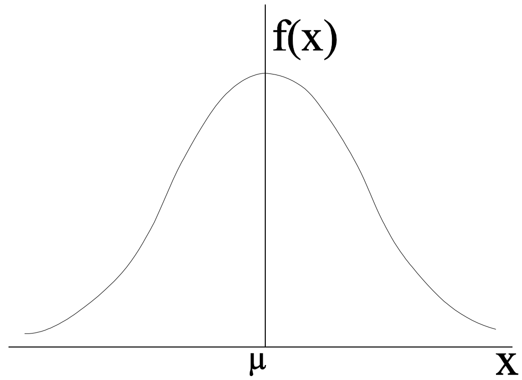
1. Normal Distributions

It is always denoted by $x \sim N(\mu, \sigma^2)$, because all characteristics of a normally distributed random variable can be represented by the first two moments, mean and variance.

The density function for a univariate is:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(x-\mu)^2}{\sigma^2} \right].$$

It's density function is symmetric about its mean. Its domain is U from -4 to +4, so it is unbounded.



2. Beta Distribution

The density function of a β-distribution is:

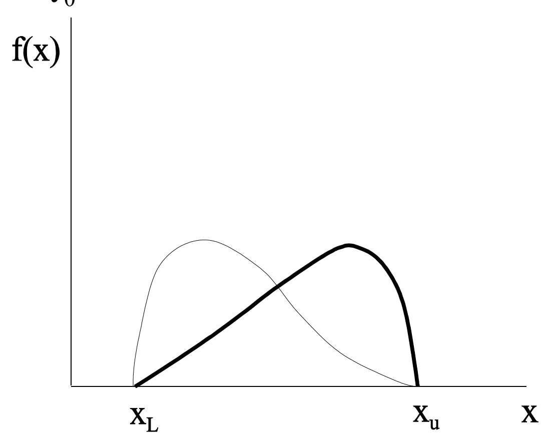
$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(x-x_L)^{\alpha-1}(x_U-x)^{\beta-1}}{x_U^{\alpha+\beta-1}}$$ for x_L # x # x_U

where x_L and x_U are the lower and upper bound of the random variable, α and β are the left and right shape parameter,

Γ( ) is the gamma function, defined as: $\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy$,

and if α is a positive integer, Γ (α) = (α-1)! – factorial.

β-distribution provides great flexibility to model real world random variables. For example, agricultural yield usually has two bounds, the lower bound is 0. Yield is also not so symmetric, skewed to the right or left.



The first three moments are

mean: $E(x) = x_L + \frac{x_U - x_L}{1 + \beta/\alpha}$

variance: $V(x) = \frac{\alpha\beta(x_U - x_L)^2}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Skewness: $S(x) = \frac{2\alpha\beta(\beta-\alpha)(x_U-x_L)^3}{(\alpha+\beta)^3(\alpha+\beta+1)(\alpha+\beta+2)} \cdot \frac{1}{\sqrt{V(x)}}$

We can see that when the two shape parameters $\alpha = \beta$ $S(x) = 0$, the distribution is then symmetric. The direction of skewness depends on the relative level of α and β. If $\alpha > \beta$, $S(x) < 0$, and if $\alpha < \beta$, $S(x) > 0$.

Reference

Rothschild, M. and J. E. Stiglitz "Increasing Risk: I. A Definition." *Journal of Economic Theory*, 2(1970):225-243