AEB 6182 Lecture IX
Professor Charles Moss

Lecture IX
Increasing Risk

I. Literature Required

A. Over the next several lectures, I would like to develop the notion of stochastic dominance with respect to a function. Meyer is the primary contributor to the basic literature, so the primary readings will be:

1. Meyer, Jack. "Increasing Risk" *Journal of Economic Theory* 11(1975): 119-32.
2. Meyer, Jack. "Choice Among Distributions." *Journal of Economic Theory* 14(1977): 326-36.
3. Meyer, Jack. "Second Degree Stochastic Dominance with Respect to a Function." *International Economic Review* 18(1977): 477-87.

B. However, in working through Meyer's articles, we will need concepts from a couple of other important pieces. Specifically,

1. Diamond, Peter A. and Joseph E. Stiglitz. "Increases in Risk and in Risk Aversion." *Journal of Economic Theory* 8(1974): 337-60.
2. Pratt, J. "Risk Aversion in the Small and Large." *Econometrica* 23(1964): 122-36.
3. Rothschild, M. and Joseph E. Stiglitz. "Increasing Risk I, a Definition." *Journal of Economic Theory* 2(1970): 225-43.

II. Introduction of "Increasing Risk"

A. This paper gives a definition of increasing risk which yields an ordering in terms of riskiness over a large class of cummulative distributions than the ordering obtained using Rothschild and Stiglitz's original definition.

B. Literature Review

1. Assuming that x and y are random variables with cummulative distributions F and G, respectively. In general we will label the cummulative distributions in such a way that G will be at least as risky as F. Further, the choice among the cummulative distributions will be made on the basis of expected utility. That is F will be preferred to G by an agent with utility function u(x) if:

$$\int_0^1 u(x)\,dF(x) \geq \int_0^1 u(x)\,dG(x)$$

Also assume that the utility function is a continuous, twice differentiable function so that we can define the Arrow-Pratt risk aversion coefficient.

2. Rothschild and Stiglitz proposed three definitions of increasing risk to eliminate weaknesses associated with using variance as a measure of risk. They showed three ways of defining increasing risk:

a. G(x) is at least as risky as F(x) if F(x) is preferred or indifferent to G(x) by all risk averse agents.

b. G(x) is at least as risky as F(x) if G(x) can be obtained from F(x) by a sequence of steps which shift weight from the center of f(x) to its tails without changing the mean.
c. G(y) is at least as risky as F(x) if y is a random variable that is equal in distribution to x plus some random noise.

3. Rothschild and Stiglitz found that necessary and sufficient conditions on the cummulative distributions F(x) and G(x) for G(x) to be at least as risky as F(x) are:

$$\int_0^1 \left[G(x) - F(x)\right] dx \geq 0 \quad \forall\, y \in [0,1]$$

$$\int_0^1 \left[G(x) - F(x)\right] dx \neq 0$$

a. Rothschild and Stiglitz showed that this definition yields a partial ordering over the set of cummulative distributions in terms of riskiness. The ordering is partial in two senses:
   (i) Only cummulative distributions of the same mean can be ordered.
   (ii) Not all distributions with the same mean can be ordered.
b. Thus, a necessary, but not sufficient condition for one distribution to be riskier than another by Rothschild and Stiglitz is that there means be equal.

4. Diamond and Stiglitz extended Rothschild and Stiglitz defining increasing risk as: G(x) is at least as risky as F(x) if G(x) can be obtained from F(x) by a sequence of steps, each of which shifting weight from the center of f(x) to its tails while keeping the expectation of the utility function, u(x), constant.

IV. Definition Based on Unanimous Preference:

A. Rothschild and Stiglitz 's first definition defines G(x) to be as risky as F(x) if F(x) is preferred or indifferent to G(x) by all risk averse agents. In other words, F(x) is unanimously preferred by the class of agents known as risk averse.
   1. Restating the general principle we could say that G(x) is at least as risky as F(x) if F(x) is unanimously preferred or indifferent to G(x) by all agents who are at least as risk averse as a risk neutral agent.
   2. This restatement shifts the focus from the absolute level of risk aversion to the relative level of risk aversion. This leads to a redefinition:
   3. Definition 1: Cummulative distribution G(x) is at least as risky as cummulative distribution F(x) if there exists some agent with a strictly increasing utility function u(x) such that for all agents more risk averse than he, F(x) is preferred or indifferent to G(x).

V. A Preserving Spread

A. A mean preserving spread is a function which when added to the density function transfers weight from the center of the density function to its tails without changing the mean.

B. Formally, s(x) is a spread if

$$\int_0^1 s(x)\,dx = 0$$

$$s(x) \text{ changes sign at most twice in } (0,1)$$

$$s(x) \geq 0, \forall\, x \leq z \text{ for some } z \in (0,1)$$

$$s(x_0) > 0, \text{for some } x_0 \in *0, z)$$

Thus, the mean-preserving spread moves probability mass to the outside of the probability density function to yield a riskier investment.

C. Definition 2: Cummulative distribution G(x) is at least as risky as cummulative distribution F(x) if G(x) can be obtained from F(x) by a finite sequence of cummulative distributions $F(x)=F_1(x),F_2(x),\ldots=G(x)$ where each $F_i(x)$ differs from $F_{i-1}(x)$ by a single spread.

VI. Implications of Definitions

A. *Theorem 1.* Consider cummulative distributions F(x) and G(x), then there is an increasing continuous function r(x) such that

$$\int_0^y \left[G(x) - F(x)\right] dr(x) \geq 0, \quad \forall y \in [0,1],$$

if and only if G(x) is at least as risky as F(x) by Definition 1.

1. Proof: Assume there exists an increasing twice differentiable r(x) such that

$$\int_0^y \left[G(x) - F(x)\right] dr(x) \geq 0, \quad \forall y \in [0,1].$$

Let z=r(x) and define functions $G^*(.)$ and $F^*(.)$ by $G^*(z)= G(r^{-1}(z))$ and $F^*(z)=F(r^{-1}(x))$. Then,

$$\int_0^y \left[G^*(r(x)) - F^*(r(x))\right] dr(x) \geq 0, \quad \forall y \in [0,1]$$

or

$$\int_{r(0)}^{r(y)} \left[G^*(z) - F^*(z)\right] dz \geq 0, \quad \forall y \in [0,1]$$

We know by previous proof (here attributed to Hadar and Russell) that

$$\int_{r(0)}^{r(y)} \left[G^*(z) - F^*(z)\right] u'(z) dz \geq 0$$

$$\Rightarrow \int_{r(0)}^{r(y)} \left[G^*(z) - F^*(z)\right] du(z) \geq 0$$

A couple of notes on this point:

a. If you made the assumption that r(x) is a one-to-one mapping, we have simply changed the definition of the cummulative distribution, defining it on the variable z which is related to the original variable x by z=r(x). This assumption would be implied by the imposition $x=r^{-1}(z)$.

b. Along the same lines, we really haven't changed the bounds of integration. We have only mapped them into the variable z.
Mapping the transformation back to x by v(x)=u(r(x)), the integral becomes

$$\int_0^1 \left[G(x) - F(x)\right] dv(x) \geq 0$$

Integrating this relationship by parts yields

$$\int_0^1 v(x)dF(x) \geq \int_0^1 v(x)dG(x)$$

for all such r(x). Next, we use the result from Pratt to show that this inequality holds for all utility functions more risk averse that r(x). A brief review of Pratt:

(i.) This article titled "Risk Aversion in the Small and Large" appeared in *Econometrica* 32(1964) pp122-36 and derives the definition of what Pratt refers to as local risk aversion:

$$r(x) = -\frac{u''(x)}{u'(x)}$$

This definition is based off the definition of the risk premium π. In this article the risk premium is defined as a function of initial wealth x. Thus, as we have come to know and love the risk premium is defined as that certain amount π(x,z) where z is a random event such that

$$u(x + E(z) - p(x,z)) = E\{u(x+z)\}$$

(ii.) Letting E(z) go to zero (or letting the random investment be actuarially neutral and taking the second order Taylor series expansion of both sides yields:

$$u(x) - p\,u'(x) + o(p^3) = E\{u(x) + zu'(x) + \tfrac{1}{2}z^2u''(x) + o(z^3)\}$$

$$u(x) - p\,u'(x) + o(p^3) = u(x) + \tfrac{1}{2}s_z^2u''(x) + o(z^3)$$

Assuming that the two second order terms go to zero for a small gamble and rearranging yields

$$p(x,z) = -\tfrac{1}{2}\frac{u''(x)}{u'(x)}s_z^2$$

$$= \tfrac{1}{2}r(x)s_z^2 \quad \text{given} \quad r(x) = -\frac{u''(x)}{u'(x)}$$

(iii.) Theorem 1 (which is used by Meyer) then states that

$$r_1(x) \geq r_2(x) \Rightarrow \begin{cases} p_1(x,z) \geq p_2(x,z) \\ u_1(u_2^{-1}(t)) \text{ is strictly convex at t} \\ \dfrac{u_1(y) - u_1(x)}{u_1(w) - u_1(v)} \leq \dfrac{u_2(y) - u_2(x)}{u_2(y) - u_2(v)} \text{ for all } v, w, x, y \text{ with } v < w \leq x < y \end{cases}$$

The point of the Pratt theorem is that for any individual more risk averse than r(x) the inequality will also hold. Thus, F(x) will be preferred by all agents who are more risk averse than the index individual.

B. *Theorem 2*. Consider cummulative distribution functions F(x) and G(x), such that their probability functions cross a finite number of times, then there is an increasing continuous twice differentiable function r(x), such that

$$\int_0^y \left[G(x) - F(x)\right] dr(x) \geq 0, \; \forall \; y \in [0,1],$$

if and only if G(x) is at least as risky as F(x) by Definition 2.

C. Next Meyer shows that his proposed definition of increasing risk yields a partial ordering over the set of cummulative distributions. In order for the definition to provide a partial ordering, it is necessary to show that the definition is (1) binary, (2) transitive, (3) reflexive, and (4) antisymetric.

1. Defining the relationship that G is at least as risky as F as $F \succ_r G$. To show transitivity then requires $F \succ_r G$ and $G \succ_r H$ implies that $F \succ_r H$. This proof is completed by defining U(r(x)) as the set of all utility functions such that u(x) is more risk averse than r(x). Given this definition, $F \succ_r G$ by all individuals such that $u(x)=U(r_1(x))$ and $G \succ_r H$ by all individuals such that $u(x)=U(r_2(x))$. Given these two definitions, we want to derive $r_3(x)$ as the level of risk aversion such that $U(r_3(x))=U(r_1(x))$˜ $U(r_2(x))$. This implies that $r_3(x)$ is that set of individuals who are both more risk averse than $r_1(x)$ and $r_2(x)$. Which implies that:

$$-\frac{r_3''(x)}{r_3'(x)} = \max\left(-\frac{r_1''(x)}{r_1'(x)}, -\frac{r_2''(x)}{r_2'(x)}\right)$$

Given that expected utility is transitive for economic agents, this result is sufficient to show that individuals who prefer G to H and F to G also prefer F to H.

2. To show that the dominance relationship is antisymmetric, we note that *Theorem 3*. $F <_r G$ and $G <_r F$ if and only if F=G.
The proof of this theorem is apparent by the integral definition. $F <_r G$ requires that

$$\int_0^y \left[G(x) - F(x)\right] d\, r_1(x) \geq 0 \; \forall y \in [0,1]$$

Similarly, $G <_r F$ would require that

$$\int_0^y \left[F(x) - G(x)\right] d\, r_2(x) \geq 0 \; \forall y \in [0,1]$$

Again defining $r_3(x)$ as that of risk averse agents who are more risk averse than both $r_1(x)$ and $r_2(x)$ yields

$$-\frac{r_3''(x)}{r_3'(x)} = \max\left(-\frac{r_1''(x)}{r_1'(x)}, -\frac{r_2''(x)}{r_2'(x)}\right)$$

Thus, for some group of risk averse agents

$$\int_0^y \left[G(x) - F(x)\right] d\, r_3(x) \geq 0 \ \forall y \in [0,1],$$

and

$$\int_0^y \left[G(x) - F(x)\right] d\, r_3(x) \geq 0 \ \forall y \in [0,1]$$

It is clear that the only way to meet both of these conditions is for G=F for all x.

4. *Theorem 4*. For any two cummulative distributions F(x) and G(x) such that their probability functions cross a finite number of times there exists an increasing continuous twice differentiable function, r(x), such that

$$\int_0^y \left[G(x) - F(x)\right] d\, r(x) \geq 0 \quad \forall y \in [0,1]$$