

## ANALYSIS OF THE EXTENDED MODEL

The Reversal Rule says that Candidate 1 wins if his observed margin  $m$  is greater than  $T$  or is between  $-T$  and 0. If  $T = 0$ , we have the conventional victory rule. This rule makes Candidate 1's payoff

$$\begin{aligned}
 \pi_1 &= \text{Prob}(x + N < -T)(0) + \\
 &\quad \text{Prob}(-T < x + N < 0)(1) + \text{Prob}(0 < x + N < T)(0) \\
 &\quad + \text{Prob}(x + N > T)(1) - c_1 \\
 &= 0 + \text{Prob}(-T - N < x < -N) + 0 + \text{Prob}(x + N > T) - c_1 \quad (2) \\
 &= [\text{Prob}(x < -N) - \text{Prob}(x < -T - N)] \\
 &\quad + [1 - \text{Prob}(x < T - N)] - c_1 \\
 &= F(-N) - F(-N - T) + 1 - F(T - N) - c_1
 \end{aligned}$$

The first order condition for Candidate 1 is

$$\frac{d\pi_1}{dc_1} = -f(-N)h' + f(-N - T)h' + f(T - N)h' - 1 = 0 \quad (3)$$

## SHOWING THAT $T \geq N$

The first order condition for Candidate 1 is

$$\frac{d\pi_1}{dc_1} = -f(-N)h' + f(-N - T)h' + f(T - N)h' - 1 = 0 \quad (4)$$

Suppose we start with  $T = 0$ . That generates an optimal value of  $c_1$  which generates a level of  $N$  that we will denote as  $N_0$ .

Now let  $T = N_0$ . That generates a new level of  $c_1$ , and a new level of  $N$  which we will call  $N_1$ .

Is  $N_1 < N_0$ ?

Yes. Here is why. The first three terms of the first order condition are the marginal benefit of spending more. The first term unchanged when  $T$  rises to  $N_0$ . The second term falls, because of unimodality of  $F$ — which means that the density is falling further from the mean, and  $f(-N - T) < f(-N)$  for positive  $T$ . The third term also falls, from  $f(-N)$  to  $f(0)$ , which is zero. Hence, the marginal benefit of buying votes falls, and the number of illegal votes falls.

This iterative process can be continued. Now set  $T = N_1$ . This will lead to a new  $N$ ,  $N_2$ , which is less than  $N_1$ , and hence less than our new  $T$ .

This process cannot continue to  $T = N = 0$ , because we know that if  $T = 0$  then  $N > 0$ . At some point, if  $T$  is reduced a little more, the first-order condition cannot be used to determine the change in  $N$ , because instead of reducing  $N$  slightly, the candidate will find it more profitable to let it jump all the way up to  $N_0$ .

Thus, either the optimal threshold converges so that  $T^* = N^*$ , or society stops at some point such that  $T^* > N^*$  so as to avoid a discontinuous jump in  $N^*$ .

## THE MODEL WITH BOTH CANDIDATES BUYING ILLEGAL VOTES

For Candidate 2,

$$\begin{aligned}
 \pi_1 &= 1 - \text{Prob}(\text{Candidate 1 wins})(1) - c_2 \\
 &= 1 - [F(-N) - F(-N - T) + 1 - F(T - N)] - c_2 \quad (5) \\
 &= -F(-N) + F(-N - T) + F(T - N) - c_2
 \end{aligned}$$

For Candidate 2, increasing  $c_2$  reduces  $N$ , so his first order condition is

$$\frac{d\pi_2}{dc_2} = -f(-N)\alpha h' + f(-N - T)\alpha h' + f(T - N)\alpha h' - 1 = 0 \quad (6)$$

The first order condition for Candidate 1 was

$$\frac{d\pi_1}{dc_1} = -f(-N)h' + f(-N - T)h' + f(T - N)h' - 1 = 0 \quad (7)$$

Thus, if  $\alpha = 1$ , so the two candidates are equally effective at buying votes, the first order conditions are identical, even if  $x_0 \neq 0$  and  $\underline{N} = 0$ , which is to say, even if one candidate has a legitimate-vote advantage or one starts with an advantage in “free” illegitimate votes.

## Implications

**That a candidate starts with a fixed advantage in legitimate or illegitimate votes does not affect the NET amount of illegitimate votes bought.**

Which candidate starts with an advantage does not affect the fact that the Reversal Rule reduces the net amount of vote buying. It does, however, affect the optimality of the Reversal Rule. If the candidate with a legitimate advantage steals fewer votes (whether for free or by buying them), then the Reversal Rule is not optimal. This is the lesson of Assumption A1.

The importance of this is that we would not expect  $N > 0$  to be correlated with Assumption A1 failing on account of Candidate 2 starting with a legitimate advantage:  $x_0 < 0$ .