A Theory of Rivalry: Does Number Two Try Harder?

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Abstract

Rivalry occurs when one player exerts effort to improve or maintain his standing relative to another player. In the model of this paper, a player can be either behind, even with, or ahead of his rival, and effort stochastically improves his position. In a one-period game, both players exert the same effort, exerting more if they are even than otherwise. In all but the last period of a T-period game, the player that is ahead exerts more effort, and in any period both players exert more if they are even than if one is ahead. Applications to innovation, elections, wars, arms races, and advertising are suggested.

I abandoned this paper because I thought that too many other similar papers were being published, and mine would not add much value.

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1. INTRODUCTION.

Avis rent a car's motto was ——. Is it true that a follower tries harder than a leader— what we might call the Avis Question? Intuition suggests several plausible, but incompatible, answers: the follower gets discouraged and slows down; the leader becomes smug and slows down; the lead does not make any difference. We will see that the lead does make a difference: in the model below, the leader exerts more effort than the follower, but both exert more when they are even than when one is ahead.

The concept of rivalry applies to a variety of situations. Will the country that is winning a war fight harder? If it is ahead in an arms race will it try hard to maintain its lead? Will the firm with the biggest share of the market indulge in the most advertising? Will the firm with the best product spend the most on research? Will the currently dominant bird most energetically fight for rank in the pecking order? Will the faction that currently has the president's ear exert the most effort to keep his favor? I will return to these questions towards the end of the paper, and suggest arguments applicable to all these situations to varying degrees.

The formal model laid out in the next section is a T-period game between two players contending for shares of a fixed prize that can depend both on the current state and the final state in period T. In each period the players simultaneously choose effort levels that stochastically affect the state the next period, which can take three values: either of the two players being ahead, or both being even with each other. or behind his rival in the next period. The model will be described in general terms for most of the paper, but readers may find it helpful to think of a dynamic innovation model in which two firms choose research spending each year in a battle over shares of a fixed market.

Some of the assumptions are special— the restriction to two rivals and three states, for example— but I hope the model will confirm one set of arguments for why the leader might exert more effort. A number of papers have addressed the Avis Question and come up with various answers. Following the terminology of Vickers (1986), the answer could be either Increasing Dominance (the leader tries harder, and tends to remain the leader) or Action-Reaction (the follower tries harder, and the lead tends to alternate). The papers addressing the question will be discussed below in Section 5. One overall lesson, however, is that general models are hard to construct and a great many influences are at work. Some papers confine themselves to a few states; others use a special assumption on how effort translates to achievement. Merely to answer the Avis Question for a particular model is useless; one ought to use the particular model to simply and convincingly illustrate a particular effect of rivalry.

Plan of the Paper.

1. Introduction.
2. Description of the model.
3. The game with one period.
4. The game with two or more periods.
5. Relaxing assumptions.
6. The rivalry literature.
7. Applications.

2. THE MODEL.

Two risk-neutral players, Alpha and Beta, are playing a T-period game of complete information with simultaneous moves and uncertainty. Let us adopt the point of view of Alpha. An achievement level is associated with each player, and the important state variable is Alpha's achievement lead over Beta. The lead can take three values: Alpha can be ahead (α), even (0), or behind (β). Let us denote the expected value of Alpha's payoff starting from the beginning of period t by V_t^{α}, V_t^{0}, or V_t^{β}, depending on the lead.

Alpha and Beta choose efforts a and b simultaneously each period. Alpha's lead rises with probability $f(a,b)$, falls with probability $g(a,b)$, and remains unchanged with probability $1 - f - g$. Subscripts and superscripts attached to these functions will denote the state of the game: the probability of an advance in period t when Alpha is ahead is f_t^{α}, for example. If Alpha is ahead and chance would call for the lead to increase, Alpha merely stays ahead, so any effort by Alpha in that state is intended purely to block Beta.

Unlike most models of innovation under uncertainty, this model does not specify a functional form for the function f. Instead, let us make general assumptions about the properties of f, an approach that will require more explanation in setting up the model, but which will point out which properties are important. The assumptions have two objectives: to guarantee existence of equilibrium efforts, and to isolate the incentive effects of the lead by ruling out purely technological advantages for the leader or the follower.

Since f is a probability, it takes values on the interval $[0,1]$. Let us assume that f does not depend on the state, except through the choices of a and b. Let us also assume that f is concave and increasing in Alpha's effort, and decreasing and convex in Beta's effort, so that Alpha's effort tends to increase the lead, but with diminishing returns, while Beta's effort tends to prevent that increase, also with diminishing returns. These assumptions imply that

$$\frac{\partial f}{\partial a} > 0 \quad (1a)$$
$$\frac{\partial^2 f}{\partial a^2} < 0 \quad (1b)$$
$$\frac{\partial f}{\partial b} < 0 \quad (1c)$$
$$\frac{\partial^2 f}{\partial b^2} > 0 \quad (1d)$$

Two further assumptions will help avoid extreme equilibria. First, assume that as Alpha's effort goes to zero, his effort's marginal product rises to infinity:

$$\overset{Limit}{a \rightarrow 0} \, \frac{\partial f}{\partial a} = \infty. \tag{2}$$

Assumption (2) guarantees an interior solution to the player's maximization problem, so that both players exert positive effort in equilibrium, whatever the state.[1] Note that since the players will always exert effort in equilibrium, whether or not changes in the lead would occur spontaneously when both players exert zero effort is unimportant to the model.

Second, assume that direct effects are stronger than indirect effects: when Alpha increases his effort, the effect on Alpha's own marginal product of effort is greater than on Beta's. This includes the case in which each player's marginal product of effort is independent of the other player's effort, a case which is perhaps the most central for this analysis, since it rules out a purely technological effect of the size of one player's effort on the other's effort.

$$\frac{\partial^2 f}{\partial a \partial b} \geq 0. \tag{3}$$

Let us make two additional assumptions to remove the possibility of purely technological differences between the leader and the follower or Alpha and Beta. First, assume that the function f is symmetric in a and b, so that effort does not have different effects on "offense" and "defense." Such an asymmetry would tilt the results in obvious ways. If, for example, effort were useless in defense ($\frac{\partial f}{\partial b} = 0$), one would not expect a player who was ahead to exert any effort. Let us therefore assume that

$$\frac{\partial f}{\partial a} = -\frac{\partial f}{\partial b} \text{ if } a = b. \tag{4}$$

Second, assume that the two players are identical except for differences in who is ahead. This implies that the form of the functions by which they advance is the same; i.e.,

$$f(x, y) = g(y, x) \; \forall x, y, \tag{5}$$

which implies, using our earlier assumptions, that $\frac{\partial g}{\partial b} > 0$, $\frac{\partial^2 g}{\partial b^2} < 0$, $\frac{\partial g}{\partial a} < 0$, and $\frac{\partial^2 g}{\partial a^2} > 0$.

It remains to specify the payoffs as functions of the state. Player Alpha's current benefit per period is denoted by R^α, R^0, or R^β. Assume that the total benefit each period is constant at $\bar{R}$, so that if Alpha is ahead, for example, Beta's current benefit is $\bar{R} - R^\alpha$. Since the players are identical, $R^0 = \bar{R}/2$, from which it follows that $R^\alpha - R^0 = R^0 - R^\beta$ (from $R^0 - R^\beta = R^0 - (\bar{R} - R^\alpha) = R^0 - 2R^0 + R^\alpha = R^\alpha - R^0$.) The assumption of constant total benefit isolates the effect of rivalry, and excludes consideration of the kind of effort a lone player might make. The game is zero-sum on the benefit side, though not on the cost side. This is a simplifying assumption; adding socially productive effort would not change the model significantly.[1]

At the end of the game there is a final balloon benefit of W^α, W^0, or W^β to Alpha, and a balloon benefit of $\bar{W}$ less Alpha's benefit for Beta. The amount W^α, for example, represents the value of being the permanent leader when the lead freezes after period T. It would take the value $\frac{\delta R^\alpha}{1-\delta}$ if the flow of benefits continues except for the absence of changes in the lead, and it would take the value R^α if the benefits simply ended. Because the firms are identical, $W^0 = \bar{W}/2$, and $W^\alpha - W^0 = W^0 - W^\beta$.

The model includes as special cases: (i) No balloon benefit ($\overline{W} = 0$); (ii) The only benefit being the balloon benefit ($\overline{R} = 0$); and (iii) winner-take-all benefits ($R^\alpha = \overline{R}$ and/or $W^\alpha = \overline{W}$).[2]

[1] xxx Lemma 3 may require removing the balloon benfit. I hope not.
[2] xxx NOt written parallel right now.

Let us denote the discount factor by $\delta > 0$, which can be either less than or equal to one, and assume that utility is separable in benefit and effort, linear in effort, and with a cost of effort normalized to unity. Linearity is purely for simplicity, and the analysis would be much the same if the marginal disutility of effort were increasing.

Alpha's value function, representing his discounted equilibrium expected payoff at the beginning of the period, is one of three equations depending on the lead. Each equation is made up of the current benefit, the current cost of effort, and the discounted expected value at the end of the period. If Alpha is ahead at the start of period t, for example, he knows that he will lose the lead with probability g and keep it with probability $1 - g$, so his value function is

$$V_t^{\alpha} = R^{\alpha} - a_t^{\alpha} + \delta g_t^{\alpha} V_{t+1}^{0} + \delta \left(1 - g_t^{\alpha}\right) V_{t+1}^{\alpha}. \tag{6}$$

The Equilibrium Concept.

As is standard in dynamic games of complete information, an equilibrium will be defined here as a subgame perfect strategy combination; i.e., the equilibrium strategies for the entire game must be Nash strategies for each subgame. Perfectness rules out non-credible threats such as Alpha threatening to exert very high effort for the rest of the game if Beta ever exerts positive effort.

Let us also restrict our attention to symmetric equilibria, requiring that the equilibrium strategies be the same for the two players. This does not mean that the players take the same actions in a given period. If Alpha is ahead in period t, for example, he might not exert the same effort as Beta, but he does exert the same effort that Beta would were Beta ahead in period t.

Symmetry of the equilibrium strategies will require some care in the proofs. We cannot impose the equations in (7) before finding the first order conditions, because symmetry of the strategies applies to the equilibrium values of the strategies, not just to any values.

For the equilibrium values of the strategies, symmetry requires that

$$a_t^0 = b_t^0, a_t^\alpha = b_t^\beta, \text{ and } a_t^\beta = b_t^\alpha. \tag{7}$$

The symmetry assumption is equivalent to finding all the subgame perfect equilibria and then discarding the asymmetric ones. Its purpose is to try to exclude equilibria whose properties have more to do with the expectations of players about each others' strategies than with fundamentals such as payoffs and initial positions. It may be that asymmetric equilibria do exist, although I have no evidence that they do.

THE GAME WITH ONE PERIOD.

The time subscript will be dropped in this section since there is only one period. Remember, however, that the one-period game is identical to the last subgame of the two-period or T-period game, a fact that will be useful later.

The value function of equation (6) becomes

$$V^{\alpha} = R^{\alpha} - a^{\alpha} + \delta g^{\alpha} W^{0} + \delta \left(1 - g^{\alpha}\right) W^{\alpha}, \tag{8}$$

and the value functions in the other two states are

$$V^{\beta} = R^{\beta} - a^{\beta} + \delta f^{\beta} W^{0} + \delta \left(1 - f^{\beta}\right) W^{\beta} \tag{9}$$

and

$$V^{0} = R^{0} - a^{0} + \delta f^{0} W^{\alpha} + \delta g^{0} W^{\beta} + \delta \left(1 - f^{0} - g^{0}\right) W^{0}. \tag{10}$$

Since $W^0 - W^\beta = W^\alpha - W^0$, equations (8) through (10) can be rewritten as

$$V^{\alpha} = R^{\alpha} - a^{\alpha} - \delta g^{\alpha}(W^{\alpha} - W^{0}) + \delta W^{\alpha}, \tag{11}$$

$$V^{\beta} = \overline{R} - R^{\alpha} - a^{\beta} + \delta f^{\beta}(W^{\alpha} - W^{0}) + \delta W^{\beta}, \tag{12}$$

and

$$V^{0} = R^{0} - a^{0} + \delta f^{0}(W^{\alpha} - W^{0}) - \delta g^{0}(W^{\alpha} - W^{0}) + \delta W^{0}. \tag{13}$$

Our assumptions on f and g guarantee that an interior solution to Alpha's problem exists. The maximands (11) to (13) are concave because f and $-g$ are concave in a, so the first order conditions characterize the optimum.

$$\frac{\partial V^{\alpha}}{\partial a^{\alpha}} = -1 - \delta \frac{\partial g}{\partial a^{\alpha}}(W^{\alpha} - W^{0}) = 0. \tag{14}$$

$$\frac{\partial V^{\beta}}{\partial a^{\beta}} = -1 + \delta \frac{\partial f}{\partial a^{\beta}}(W^{\alpha} - W^{0}) = 0. \tag{15}$$

$$\frac{\partial V^0}{\partial a^0} = -1 + \delta \frac{\partial f}{\partial a^0}(W^\alpha - W^0) - \delta \frac{\partial g}{\partial a^0}(W^\alpha - W^0) = 0. \qquad (16)$$

Note first that the assumptions of the model guarantee existence of an equilibrium in pure strategies.

Lemma 1: In a one-period model or the last period of a T-period model there exists a Nash equilibrium in pure strategies in which each player exerts positive effort.

Proof:
1. By a theorem from my book, pure-strategy equilibrium exists if the strategy sets are compact, and payoff functions are continuous in strategies of both players and quasi-concave in own-strategies.
2. If we put an upper bound on effort, the strategy sets are compact. Payoffs are continous, and are concave in own-effort.
3. But in fact if the bound is high engouh, the solution is interior.
4. And if th solutin is interior, the bound can be removed without makin any diffrence.

Proposition 1: *In a one-period model or the last period of a T-period model, when one player is ahead his effort is identical to the other player's effort and less than the effort when the two players are even with each other, i.e.*

$$a^\alpha = a^\beta \tag{17}$$

and

$$a^\alpha < a^0. \tag{18}$$

Proof: From first order conditions (14) and (15),

$$-\frac{\partial g}{\partial a^\alpha} = \frac{\partial f}{\partial a^\beta}. \tag{19}$$

Suppose that Beta has the same effort when behind as when ahead, so that $b^\alpha = b^\beta$. Then assumption (4) tells us that equality (19) is true only if $a^\alpha = a^\beta$. Since Beta has first order conditions corresponding to Alpha's, it is indeed rational for Beta to choose $b^\alpha = b^\beta$ in a Nash equilibrium, and equality (17) is proved.

To prove inequality (18), note from (15) and (16) that

$$\frac{\partial f}{\partial a^0} - \frac{\partial g}{\partial a^0} = \frac{\partial f}{\partial a^\beta}. \tag{20}$$

Given that $\frac{\partial g}{\partial a} < 0$, equation (20) implies that

$$\frac{\partial f}{\partial a^0} < \frac{\partial f}{\partial a^\beta}. \tag{21}$$

Given assumption (3) and the fact that $a^0 = b^0$ by symmetry, it follows from concavity of f in a that $a^0 > a^\beta = a^\alpha$.
Q.E.D.

Proposition 1 is interesting even aside from its relevance to T-period games. Except for the unproductive nature of the achievement in this model, the one-period game is like a one-stage patent race under uncertainty, with the complication that the players might not start out even. When one player gains what the other loses, so they exert the same effort, and if the stakes are higher they both exert a greater effort.

Equality (17) says that if one player is ahead, his effort is the same as if he were behind, so by symmetry of the equilibrium strategies, the follower exerts the same effort as the leader. The leader and the follower have exactly the same incentives. They are fighting over the difference between being ahead and being even, the leader defending a piece of property and the follower attacking it. This is a special property of the one-period game, because in one period the follower has no chance to become the leader, the leader has no chance to become the follower, and neither need look ahead to the cost of future effort.

Inequality (18) says that both players exert more effort when they are even than otherwise. If the two players are even, their effort helps each one both defensively and offensively. Moreover, the difference between worst-case failure and best-case success is at its maximum for each player at a value of $V^{\alpha} - V^{\beta}$; more is at stake. If the players are not even, the leader's effort is purely defensive and the follower's is purely offensive, so only $V^{\alpha} - V^{0}$ is at stake.

4. THE GAME WITH TWO OR MORE PERIODS.

The two-period game is the simplest multi-period game, so let us analyze it in some detail. We must now be careful about attaching time subscripts to variables. Denote the last period, whose characteristics were described in the preceding section, by the subscript T. Before starting to find the optimal strategies in the first period of the two-period game, it is useful to know the difference between the possible values in the second period. Since this difference will recur through the rest of the analysis, let us define two new variables:

$$\begin{aligned} D_t^\alpha &\equiv V_t^\alpha - V_t^0, \\ D_t^0 &\equiv V_t^0 - V_t^\beta. \end{aligned} \tag{22}$$

We can now succintly state Lemma 2, which will not only be useful for the proof of Theorem 2, but has some independent interest, since it states that in the last period the difference between being in the lead and being even is greater than the difference between being even and being behind.

Lemma 2:

$$D_T^\alpha > D_T^0. \tag{23}$$

Proof:
Using equations (11) and (13), one obtains

$$\begin{aligned} V_T^\alpha - V_T^0 &= (R^\alpha - R^0) - (a_T^\alpha - a_T^0) - \delta g_T^\alpha (W^\alpha - W^0) - [\delta f_T^0 (W^\alpha - W^0) \\ &\quad - \delta g_T^0 (W^0 - W^\beta)] + \delta W^\alpha - \delta W^0. \end{aligned} \tag{24}$$

Since $W^\alpha - W^0 = W^0 - W^\beta$ by the symmetry assumption, and $f_T^0 = g_T^0$ by Proposition 1, equation (24) becomes

$$D_T^\alpha = (R^\alpha - R^0) - (a_T^\alpha - a_T^0) + \delta(1 - g_T^\alpha)(W^\alpha - W^0). \tag{25}$$

Using equations (11) and (12), one obtains

$$\begin{aligned} V_T^0 - V_T^\beta &= \\ &= (R^0 - \overline{R} + R^\alpha) - (a_T^0 - a_T^\beta) + \left[\delta f_T^0 (W^\alpha - W^0) - \delta g_T^0 (W^0 - W^\beta)\right] \\ &\quad - \delta f_T^\beta (W^0 - W^\beta) + \delta W^0 - \delta W^\beta. \end{aligned} \tag{26}$$

Since by Proposition 1 ($a_T^\beta = a_T^\alpha$ and $f_T^\beta = g_T^\alpha$, equation (26) becomes

$$D_T^0 = (R^\alpha - R^0) + (a_T^\alpha - a_T^0) + \delta(1 - g_T^\beta)(W^\alpha - W^0). \tag{27}$$

Since $g_T^\alpha = f_T^\beta$ by Proposition 1, and $R^\alpha - R^0 = R^0 - R^\beta$, equations (25) and (27) can be rewritten as

$$D_T^\alpha = \{(R^\alpha - R^0) + \delta(1 - g_T^\alpha)(W^\alpha - W^0)\} + \{a_T^0 - a_T^\alpha\} \tag{28}$$

and

$$D_T^0 = \{(R^\alpha - R^0) + \delta(1 - g_T^\alpha)(W^\alpha - W^0)\} - \{a_T^0 - a_T^\alpha\}. \tag{29}$$

Since Alpha's value is greater if the lead is greater, both expressions (28) and (29) are positive. By Proposition 1, $\{a_T^0 - a_T^\alpha\} > 0$. Comparing equation (28) with (29) therefore tells us that $D_T^\alpha > D_T^0$.
Q.E.D.

Now the problem of finding the equilibrium strategies in the first period can be tackled. The values of Alpha in the three possible states are

$$V_{T-1}^\alpha = R^\alpha - a_{T-1}^\alpha - \delta g_{T-1}^\alpha D_T^\alpha + \delta V_T^\alpha, \tag{30}$$

$$V_{T-1}^\beta = R^\beta - a_{T-1}^\beta + \delta f_{T-1}^\beta D_T^\beta + \delta V_T^\beta, \tag{31}$$

and

$$V_{T-1}^0 = R^0 - a_{T-1}^0 + \delta f_{T-1}^0 D_T^\alpha - \delta g_{T-1}^0 D_T^0 + \delta V_T^0. \tag{32}$$

The first order conditions are (recalling that that D_T^α and D_T^β are independent of the effort levels in the first period)

$$\frac{\partial V_{T-1}^\alpha}{\partial a_{T-1}^\alpha} = -1 - \delta \frac{\partial g}{\partial a_{T-1}^\alpha} D_T^\alpha = 0, \tag{33}$$

$$\frac{\partial V_{T-1}^\beta}{\partial a_{T-1}^\beta} = -1 + \delta \frac{\partial f}{\partial a_{T-1}^\beta} D_T^0 = 0, \tag{34}$$

and

$$\frac{\partial V_{T-1}^0}{\partial a_{T-1}^0} = -1 + \delta \frac{\partial f}{\partial a_{T-1}^0} D_T^\alpha - \delta \frac{\partial g}{\partial a_{T-1}^0} D_T^0 = 0. \tag{35}$$

These first order conditions can be used to prove Proposition 2.

Proposition 2: *In the next-to-last period each player exerts greater effort when even with the other player than if one of them is ahead. If one of them is ahead, the leader exerts greater effort than the follower. Therefore,*

$$a^{\alpha}_{T-1} < a^{0}_{T-1} \tag{36}$$

and

$$a^{\beta}_{T-1} < a^{\alpha}_{T-1}. \tag{37}$$

Proof: To prove inequality (36), note that first order conditions (33) and (35) imply that

$$-\frac{\partial g}{\partial a^{\alpha}_{T-1}} D^{\alpha}_{T} = \frac{\partial f}{\partial a^{0}_{T-1}} D^{\alpha}_{T} - \frac{\partial g}{\partial a^{0}_{T-1}} D^{0}_{T}. \tag{38}$$

Since in equilibrium $a^{0}_{T-1} = b^{0}_{T-1}$ by symmetry, assumption (4) implies that $\frac{\partial f}{\partial a^{0}_{T-1}} = -\frac{\partial g}{\partial a^{0}_{T-1}}$. Hence one may rewrite (38) as

$$-\frac{\partial g}{\partial a^{\alpha}_{T-1}} D^{\alpha}_{T} = -\frac{\partial g}{\partial a^{0}_{T-1}} (D^{\alpha}_{T} + D^{0}_{T}), \tag{39}$$

in which case

$$-\frac{\partial g}{\partial a^{\alpha}_{T-1}} > -\frac{\partial g}{\partial a^{0}_{T-1}}, \tag{40}$$

which implies, given assumption (3) that $\frac{\partial^2 f}{\partial a \partial b} \geq 0$, that $a^{0}_{T-1} > a^{\alpha}_{T-1}$.

To prove inequality (37), note that since $D^{\alpha}_{T} > D^{0}_{T}$, first order conditions (33) and (34) imply that

$$-\frac{\partial g}{\partial a^{\alpha}_{T-1}} < \frac{\partial f}{\partial a^{\beta}_{T-1}}. \tag{41}$$

By assumption (5) and the assumption that the equilibrium is symmetric, $-\frac{\partial g}{\partial a^{\alpha}_{T-1}} = -\frac{\partial f}{\partial b^{\beta}_{T-1}}$, so equation (41) implies

$$-\frac{\partial f}{\partial b^{\beta}_{T-1}} < \frac{\partial f}{\partial a^{\beta}_{T-1}}. \tag{42}$$

By assumption (4), if $a^{\beta}_{T-1} = b^{\beta}_{T-1}$ then expression (42) would be an equality. Since it is an inequality, the concavity of f in a combined with assumption (5) and symmetry tells us that $a^{\beta}_{T-1} < b^{\beta}_{T-1}$. In equilibrium, $b^{\beta}_{T-1} = a^{\alpha}_{T-1}$, so (37) is proven.
Q.E.D.

Propositions 1 and 2 show that players behave differently in the first and second periods of a game. The proof method of Proposition 2 can be applied recursively to show that it is the last period that is special, not the first; the inequalities of Proposition 2 apply to all but the last period. This is stated in Proposition 3, whose proof is the Appendix.

Proposition 3: *In all but the last period of a T-period game, each player exerts greater effort when even with the other player than if one of them is ahead. If one of them is ahead, the leader exerts greater effort than the follower. For every $t < T$,*

$$a^{\alpha}_t < a^0_t. \tag{43}$$

and

$$a^{\beta}_t < a^{\alpha}_t \tag{44}$$

Proof: See the Appendix.

Note the implications here for the amount of time spent in each state. As T gets large, we can predict that more time will be spent in states with one ahead or behind.

5. RELAXING ASSUMPTIONS.

It may not be easy to tell which assumptions are important in generating Propositions 1 to 3. In this section I will discuss the properties of the model and consider relaxing certain of the assumptions.

Intuition behind the propositions.

Section 3 explained why in the last period the leader's effort level is the same as the follower's: what one gains, the other loses. Propositions 2 and 3, however, say that in every period but the last, the leader exerts more effort. The reason is that expected future effort affects the choice of current effort. Consider a two-period game. If effort levels were always zero in the second period, then in the first period the leader and the follower would have identical incentives and exert the same effort. But effort levels in the second period are higher if the players are even than if somebody is ahead. When the follower succeeds in becoming even, he gains in benefit, but his second period effort increases too. When the leader successfully stays ahead, he both preserves his benefit level and avoids increasing his second period effort. The leader's incentive to exert effort is therefore greater than the follower's.

In every period, including the last, the players exert the most effort when they are even. In the last period, this is because when the players are even, the possible changes in position are greatest. When a player can either advance or regress, research is useful for both offense and defense. The same intuition applies in the earlier periods, but it is reinforced by the effect described in the last paragraph. Moving away from being even lowers future effort, while moving away from being the leader or the follower increases future effort.

The explicit assumptions were discussed as they arose in describing the model, but more can be said about some of the more implicit assumptions.

Three States, not N. The assumption that a player can only be ahead, behind, or even is a simplifying assumption. Although the complexity and notational difficulty increases rapidly, the intuition behind the propositions continues to hold with more states. Consider a leader whose effort can increase his share of the benefit instead of just defending it. His share cannot increase indefinitely, since it is bounded by one hundred percent, so either (a) his effort eventually becomes purely defensive or (b) the offensive component of his incentive becomes smaller as the bound is approached. In either case, there is less at stake for the leader as he pulls further ahead. Since the benefit side of the game is zero-sum, this implies that there is less at stake for the follower. If this verbal reasoning is correct, effort will be greater when the lead is closer to zero, and this implies that the leader has more incentives than the follower. [3]

One-Level Transitions, not Two-Level. An implicit assumption is that change in the lead is gradual: there is not a jump from being ahead to being behind without the transition of being even. One might wonder whether this assumption drives the result that effort is highest when the firms are even. Indeed it does, but the assumption can be relaxed considerably without losing the result. In the extreme, suppose that the function f represents not the probability of a change of one level in the lead, but rather the probability of Alpha gaining the lead. If Alpha starts out behind, f is the probability of gaining two levels, while if Alpha starts out even, f is the probability of gaining one level.[4] The first order condition for Alpha when he starts with a lead changes only in the substitution of W^β for W^0. Equation (14) is transformed to

$$\frac{\partial V^\alpha}{\partial a^\alpha} = -1 - \delta \frac{\partial g}{\partial a^\alpha}(W^\alpha - W^\beta) = 0. \tag{45}$$

But this substitution is enough that in combination with the unchanged first order condition (16) an equivalent to (20) can be generated:

$$\frac{\partial f}{\partial a^0}(W^\alpha - W^0) - \frac{\partial g}{\partial a^0}(W^\alpha - W^0) = \frac{\partial g}{\partial a^\alpha}(W^\alpha - W^\beta), \tag{46}$$

which implies, because $\frac{\partial f}{\partial a^0} = -\frac{\partial g}{\partial a^0}$ and $(W^\alpha - W^\beta) = 2(W^\alpha - W^0)$, that

$$-2\frac{\partial g}{\partial a^0}(W^\alpha - W^0) = -\frac{\partial g}{\partial a^\alpha}2(W^\alpha - W^0), \tag{47}$$

[3] xxx Cite Harris paper here.

which implies that $a^\alpha = a^0$. In this extreme case, the results in Propositions 1 to 3 disappear. In a given period the players exert the same effort regardless of the state, although that effort may vary as time passes. But an intermediate model that broke up f into a probability θf that Alpha's lead increases by one level and a probability $(1-\theta)f$ that it increases by two would restore the conclusion that $a^0 > a^\alpha$, with the size of $a^0 - a^\alpha$ depending on θ.

A Transition Functions with Three Values, not Two. Another modification is to define f to be the probability of an increase in the lead and $(1-f)$ to be the probability of a decrease. If a period started with the players even, either Alpha or Beta would be ahead at the end of the period: stasis is not allowed. All the assumptions of the original model concerning f are still required, and the results are exactly the same: Propositions 1, 2, and 3 remain true.

Being even is a transitory state in the two-step transition model. The model might start with both players even, but it never returns to that state.

T **periods, not an Infinite Number.** The assumption that the number of periods is finite is important for modelling reasons. Proposition 3 takes us most of the way to an infinite period model, because T can be great enough that the last period exerts a trivial effect. The Folk Theorem of repeated games, however, tells us that in a wide variety of games with infinite repetitions and sufficiently little discounting, a large number of equilibria exist (see Fudenberg & Maskin [1986]).

The rivalry model in this paper is not a stationary repeated game, because the game can be in any of three states. This nonstationarity means that the Folk Theorem does not apply directly. But the reasoning behind the Folk Theorem— that with sufficiently little discounting, arbitrary equilibrium behavior can be enforced by the threat of future punishments— can be made to apply. The punishment would take the form of very high effort by the punisher for a large number of periods, and if the punisher deviated by not carrying out the punishment, which is costly for him, the strategy would specify that he himself be punished by the other player. As a result of this argument, the infinite period game is not so interesting as the finite period game. The outcomes characterized in the main part of the paper can be supported by equilibria in both games, but the infinite period game is underdetermined and has other equilibria as well .

6. THE RIVALRY LITERATURE.

Outside of the literature on technical change, papers which ask questions about rivalry include J. Hirshleifer (1987), Aron & Lazear (1987), and Dixit (1987). Hirshleifer analyzes the choice between fighting and producing, a choice that is not available in the model here. Aron and Lazear ask whether firms that are leaders in market share in the current market are less likely to switch to a new product line. This is an example of rivalry, but the emphasis is not on the intensity of a single rivalrous action, but on the variety of rivalrous actions available.

Dixit (1987) examines pure rivalry in a one-period contest in which both players start at the same level, but the progress function might give one player an advantage. If the progress function is symmetric, Dixit shares the conclusion of this paper that both players exert the same effort. The present paper has ruled out asymmetric progress functions, but it it interesting to note that Dixit finds that the player with the advantage (that is, who is more likely to win if effort levels are the same) exerts greater effort. This is proved for particular functional forms, and the reason behind the result is completely different from the reason in the multi-period model of rivalry. [4]

The Innovation Literature.

Are Lippman-Mccardle and Harris-Vickers(1985) relevant? Is Beath(1987) relevant?

XW

Rivalry has been most studied in the context of innovation. Most of the literature analyzes single-innovation models, in which research ends after the innovation is found. This can describe rivalry over time if the research process has many stages (see Fudenberg, Gilbert, Stiglitz & Tirole [1983], Harris & Vickers [1985], and Lippman & McCardle [1987]), but a patent is special in the sense that one player gets it and the other player does not;

[4] xxx what is that reasons?

how close the loser comes does not matter.

The Avis Question came up first in the patent-race models of Gilbert & Newbery (1982) and Reinganum (1985). Gilbert and Newbery construct a model of sleeping patents in which payoffs may be non-zero-sum even on the benefit side. One firm has a monopoly on a technology and another threatens to discover a substitute. The monopolist does research to try to patenting the substitute first, whether he intends to use it or not, and he spends more than his rival if a monopoly with access to both technologies could earn higher profits than a duopoly. Reinganum (1985) models a sequence of patentable innovations and reaches a conclusion opposite to both Gilbert & Newbery and the present paper: the incumbent spends less on research than the rivals. This is because the time between innovations is endogenous. The expected cost of discovering the next innovation in the sequence is equal for each firm in this model, so once a particular patent race ends, no firm enjoys any advantage in discovering the next patent. The incumbent spends less on research because the net revenue flow from the new technology is the same whichever firm discovers it, but because the incumbent loses the revenue flow from the old technology, his net benefit from innovation is less. The endogeneity of the interval between discoveries is crucial: the incumbent has less or no incentive to hurry the discovery. The biggest difference from the present paper, however, is that in the Reinganum model the incumbent has no advantage over the other firms in the struggle for the future incumbency; he is just an incumbent, not a leader. Which firm makes the next innovation is thus irrelevant to the future amount of research.

The model of Vickers (1986) can be considered an application of this idea when time periods are exogenous and the innovation is not "drastic". There is a sequence of falling cost levels that firms move throguh. Two firms. T periods. There is a new lower cost level for each period, and the players bid for who gets to have that cost level. No discounting, but that does not matter. Certainty. You can jump ahead with no problem. The outcome is action-recton if industry prfits are higher when the follower's costs fall. If the high-cost firm gets zero profits, we gte increasing domiancne.

A number of recent models have looked at the Action-Reaction vs. Increasing Dominance question in more detail. These models see rivalry as progress along a sequence of achievement levels, which in an innovation model would consist of different products or technologies. The paper closest to the present paper are Grossman & Shapiro (1987), Harris & Vickers (1987), and Harris (1988).

Grossman & Shapiro (1987) look at a multi-stage patent discovery process, in which there is only one innovation to be found, which will not be found without research. The research technology is an arrival process, and the game ends once one player reaches the final stage of the patent search. Their conclusions on which firm exerts more effort are consistent with the predictions of the present paper, but the conclusions are driven by different forces. They find that rivals spend more on research when they are even for two reasons: the potential change in values is then greatest, and the reaction curves are upward sloping. The first of these reasons drives the same result in the present paper. Grossman and Shapiro also find that the leader spends more than the follower, but their reason is not the expectation of future costs, but that the leader is closer to the finish line of the patent race.

Other papers have dealt with other aspects of sequential innovation. Beath, Katsoulacos, & Ulph (1987) focus on how the speed of the innovation technology and the form of competition in the product market affect research spending to find a patent. These also include Beath, Katsoulacos, & Ulph (1989) and Aoki 88. Aoki (1988) uses the framework of a stochastic game with an infinite number of periods. The follower can earn profits only if his technology is close enough to the leader's, and there are three possible profit levels: zero, duopoly, and monopoly.

7. APPLICATIONS.

The model is general enough to apply to many situations of rivalry. I will list some such situations below, with suggestions as to what would represent effort, the lead, the current benefits, and the balloon benefit. The model has made ruthless use of the *ceteris paribus* assumption, so one would not expect it to fully explain any single situation, but it does point out effects of rivalry that are common to many situations. These effects are superimposed on other tendencies that may reverse the predictions of the model. The model itself has different predictions depending on whether the situation is best modelled as a one period game or as a multiperiod game. In the discussion below I will treat the situations as multiperiod games, which assumes that the players have a chance to revise their effort levels after observing the effect of their earlier efforts.

(1) Innovation. In an innovative industry, a firm's market share depends on the quality of its product relative to its competitor's product. To maintain relative quality, a firm must spend on research, but success in research is uncertain, and a follower firm does have a chance to catch up to the leader. In some cases the benefit will be the profits from current production, but in other cases there may be an additional balloon benefit if at some point the leading firm becomes able to maintain its sales even if it slackens it research; e.g. if there are network externalities. The rivalry model predicts that research spending will be highest when two firms split the market evenly, and that if one firm is ahead, that firm will spend the most on research.

(2) Wars. In a war, two nations exert effort to capture a resource whose value does not increase if they fight more intensely, and may even decrease. The lead is the probability of winning a complete victory, and effort is the amount of resources spent on the war. The benefits of being ahead are likely to be small while the war continues, but at the end the country which is ahead can acquire territory, influence over the government of the vanquished, or influence over third countries. A difference from the rivalry model is that the end date of a war is endogenous. The model predicts that a nation that is winning will fight harder than a nation that is losing, because the losing nation's successes in individual battles prolong the suffering of war. The effort exerted will be most intense when neither side is clearly winning.

(3) Arms Races. In an arms race, two countries strive to increase their influence by military superiority that they may never directly use. The difference from a war is that the cost is purely in dollars rather than lives lost, and that an arms race is less likely to end. One form of arms race— the race to have the best arms technology by spending on research— is practically identical to the rivalry model. What is more generally thought of as an arms race— spending on the stock of weapons— fits the rivalry model if it is uncertain how this spending translates into gains in power.

The lead in this second kind of arms race is the likelihood of winning a war, should one break out, and the effort is the amount spent on arms. The current payoffs are the amounts of influence derived from the threat to start and win a war, and the final period can be far enough away that the balloon payoff is unimportant. The rivalry model's prediction is that the nation which is winning the arms race will spend more than the loser, because the loser's successes increase future spending on arms. Both nations will spend more when their arms levels are close than when the gap is large. Note, however, that implicit in the rivalry model is the stationarity of the functions translating effort into gains. If accumulating more than a certain number of warheads, for example, fails to increase a nation's power, the model's assumptions are invalid and one would not expect the leader to accumulate more than that number.

(4) Advertising. The rivalry model fits advertising in the same way as it fits innovation, since what is important about a product is its profitability, whether that is derived from technological excellence or consumer perceptions. Although the assumption of a market of constant size is not crucial to the model, it does seem more applicable to an advertising model than an innovation model. The lead is a measure of buyer preference, and effort is spending on advertising. The model predicts that the firm with the biggest market share will spend most heavily on advertising, and that both firms will spend more when their market shares are equal.

(5) Pecking Orders. Humans and lower animals exert effort to maintain and improve their positions in pecking orders. A pecking order can be useful to a social group, but the benefit to an individual who moves up is largely a loss to the individual who moves down. The lead is position in the pecking order, and effort could be anything from physical combat to achievements that benefit and impress the rest of the group. The rivalry model predicts that close rivals would exert more effort, and that the dominant individual would be the more willing to prolong a power struggle.

The rivalry model also has implications for why a species might benefit from (a) dominance hierarchies and (b) the emotion of demoralization. Dominance hierarchies are useful because they diminish the wasteful expenditure of effort that the rivalry model implies would occur in a group of equals. Demoralization is helpful because the loser in a struggle is benefited by diminishing his effort, whether he diminishes it for rational or emotional reasons. If the rivalry model is correct, one would therefore expect evolutionary pressures to lead to hierarchies and demoralization.

(6) Internal Politics. Suppose that each member of an organization lends some amount of support to each of the two factions into which it is divided. Applying the rivalry model, the lead can be viewed as the amount by which a given member (most importantly, the organization's leader) supports one faction, and effort is the amount of time and resources spent trying to persuade him to change his support. The model predicts that the faction currently controlling a member is more willing to exert effort to keep him, but that even more effort would be spent if he were undecided.

(7) Elections. A reasonable simplification is that an election campaign yields no current benefits to the two candidates, but on election day whoever is ahead is paid a balloon benefit. Ties are unlikely, so the model with the two-valued transition function fits the situation best. The model predicts that campaign spending will be highest when the candidates are running even in the polls, and that if one candidate is ahead, he will spend more than his lagging rival.

(8) Artificial Tournaments. A tournament is an incentive mechanism that distributed rewards according to relative performance instead of absolute performance, for performing best rather than for performing well. From the work of Lazear & Rosen (1981), Nalebuff & Stiglitz (1983), and others, it is well-known that principals can find tournaments helpful in overcoming agency problems. Examples include awarding a profitable contract to the aircraft company that comes up with the best jet fighter design by a certain date, giving tenure to the assistant professor who has the best research output in the department, and presenting a company award to the industrial plant with the best output or safety record during a particular month.

It has already been established that a tournament will not work well if the contestants differ too much in ability. The rivalry model adds a dynamic counterpart to that idea: the tournament will induce the most effort if the contestants happen to stay even with each other over the time period in which the tournament takes place. Whether the designer of a tournament can make use of this knowledge is a question too involved to try to answer here.

(9) Rentseeking. Rentseeking is halfway between the economic and military examples: economic agents expend effort to induce the government, with its monopoly of force, to change the economic system in their favor. Coalitions are often important in the political process of rentseeking (see, for example, Gilligan et al. [forthcoming]), but sometimes rentseeking is simple rivalry between firms bidding for government help against consumers, or buyers and sellers bidding for government help against each other.

The rivalry model implies that if one group is succeeding in capturing a government body, then it will exert more effort than its competitor. Suppose, for example, that the opinion of the chairman of a congressional transportation committee is affected both by campaign contributions and by random factors such as the merits of an issue. According to the model, both the railroad and the trucking trade associations would always contribute to him, but if he has become pro-railroad, the railroads would contribute more. If the congressman were undecided, he would attract still greater contributions (though in this model his stance is endogenous, so he cannot choose to be undecided).

Concluding Remarks.

The model has shown that under the assumptions of pure rivalry, in most periods the leader exerts more effort than the follower, but both exert less effort than if they were even with each other. An exception is the last period, or the only period of a one-period model, in which the leader and the follower exert the same effort.

Each part of the result has an intuitive explanation. The leader's success reduces the cost of future competition, while the follower's success increases it, so the leader has more incentive to exert effort. When the rivals are even, the effect of a change is greater, so both players have more incentive. And in the last period, what one player gains, the other loses, so the incentives are evenly balanced.

Appendix.

The appendix contains the proof of Proposition 3. It is convenient to first prove a lemma regarding different states' valuations in the two-period game.

This next lemma has a flawed proof, and it is crucial. The problem is that in period T-1 the big thing is to avoid high costs in period T; which is to avoid being in state 0 at T. But maybe starting in state 0 at T-1 is the best ...

The key thing is to show that the probability of remaining in an extreme state is greater than the probability of going to an extreme state from the 0 state. And that, I think , is impossible to rpovie. Alternately, and easier, get rid of the balloon payment, and then we can prove that spending is no greataer in the last period.

Lemma 3: *If* $D_t^\alpha > D_t^0$, *then* $D_{t-1}^\alpha > D_{t-1}^0$.

Proof: Generalizing equations (30) to (32) from a two-period model to an n-period model, we obtain

$$V_t^\alpha = R^\alpha - a_t^\alpha - \delta g_t^\alpha D_{t+1}^\alpha + \delta V_{t+1}^\alpha, \tag{48}$$

$$V_t^\beta = R^\beta - a_t^\beta + \delta f_t^\beta D_{t+1}^0 + \delta V_{t+1}^\beta, \tag{49}$$

and

$$V_t^0 = R^0 - a_t^0 + \delta f_t^0 D_{t+1}^\alpha - \delta g_t^0 (D_{t+1}^0 + \delta V_{t+1}^0. \tag{50}$$

Differentiating, one obtains the first order conditions

$$\frac{\partial V_t^\alpha}{\partial a_t^\alpha} = -1 - \delta \frac{\partial g}{\partial a_t^\alpha} D_{t+1}^\alpha = 0, \tag{51}$$

$$\frac{\partial V_t^\beta}{\partial a_t^\beta} = -1 + \delta \frac{\partial f}{\partial a_t^\beta} D_{t+1}^0 = 0, \tag{52}$$

and

$$\frac{\partial V_t^0}{\partial a_t^0} = -1 + \delta \frac{\partial f}{\partial a_t^0} D_{t+1}^\alpha - \delta \frac{\partial g}{\partial a_t^0} D_{t+1}^0 = 0. \tag{53}$$

Using equations (48) to (50), one obtains

$$\begin{aligned} D_{t-1}^\alpha &= R^\alpha - R^0 - (a_{t-1}^\alpha - a_{t-1}^0) - \delta g_{t-1}^\alpha D_t^\alpha + \delta V_t^\alpha - \delta f_{t-1}^0 D_t^\alpha + \delta g_{t-1}^0 D_t^0 - \delta V_t^0, \\ &= R^\alpha - R^0 + (a_{t-1}^0 - a_{t-1}^\alpha) + \delta(1 - g_{t-1}^\alpha) D_t^\alpha - \delta f_{t-1}^0 D_t^\alpha + \delta g_{t-1}^0 D_t^0 \end{aligned} \tag{54}$$

and

$$\begin{aligned} D_{t-1}^0 = \ & R^0 - R^\beta - (a_{t-1}^0 - a_{t-1}^\beta) + \delta f_{t-1}^0 D_t^\alpha - \delta g_{t-1}^0 D_t^0 + \delta V_t^0 - \delta f_{t-1}^\beta D_t^0 - \delta V_t^\beta, \\ &= R^0 - R^\beta - (a_{t-1}^0 - a_{t-1}^\beta) + \delta(1 - f_{t-1}^\beta) D_t^0 - \delta g_{t-1}^0 D_t^0 + \delta f_{t-1}^0 D_t^\alpha. \end{aligned} \tag{55}$$

Using equations (54) and (55):

$$\begin{aligned} D_{t-1}^\alpha - D_{t-1}^0 = \ & 2a_{t-1}^0 - a_{t-1}^\alpha - a_{t-1}^\beta \\ & + \delta\{(1 - g_{t-1}^\alpha) D_t^\alpha - f_{t-1}^0 D_t^\alpha + g_{t-1}^0 D_t^0 - (1 - f_{t-1}^\beta) D_t^0 + g_{t-1}^0 D_t^0 - f_{t-1}^0 D_t^\alpha\}. \end{aligned} \tag{56}$$

By symmetry of the equilibrium, $g_{t-1}^\alpha = f_{t-1}^\beta$ and $f_{t-1}^0 = g_{t-1}^0$, so we can rewrite equation (56) as

$$D_{t-1}^\alpha - D_{t-1}^0 = \{2a_{t-1}^0 - a_{t-1}^\alpha - a_{t-1}^\beta\} + \delta\{(1 - g_{t-1}^\alpha - 2g_{t-1}^0)(D_t^\alpha - D_t^0)\}. \tag{57}$$

Because probabilities must add up to one, and because Proposition 1 tells us that $f_{t-1}^\alpha > g_{t-1}^\alpha$ and $g_{t-1}^0) = f_{t-1}^0)$, we know that g_{t-1}^α and $g_{t-1}^0)$ are each no greater than 0.5 in equilibrium. Hence, $\delta(1 - g_{t-1}^\alpha - 2g_{t-1}^0) \in [-.5, 1]$. If that expression is positive, then expression (57) is positive and the proposition is proved. If it is negative, then the proposition is proved if it can be shown that

$$(D_t^\alpha - D_t^0) \leq \{2a_{t-1}^0 - a_{t-1}^\alpha - a_{t-1}^\beta\}. \tag{58}$$

We can prove Lemma 3 for $t =$, because for $t = T$ it turns out that

$$D_T^\alpha - D_T^0 = 2a_T^0 - a_T^\alpha - a_T^\beta. \tag{59}$$

By substituting a value taken from first order condition (52) for the 1 in first order condition (53), and then dividing by δ, we obtain

$$-\frac{\partial f}{\partial a_t^\beta} D_{t+1}^0 + \frac{\partial f}{\partial a_t^0} D_{t+1}^\alpha - \frac{\partial g}{\partial a_t^0} D_{t+1}^0 = 0. \tag{60}$$

Recognizing that by symmetry $-\frac{\partial g}{\partial a_t^0} = \frac{\partial f}{\partial a_t^0}$, equation (60 becomes

$$\frac{\partial f}{\partial a_t^\beta} D_{t+1}^0 = +\frac{\partial f}{\partial a_t^0}(D_{t+1}^\alpha + D_{t+1}^0). \tag{61}$$

With a little rearranging, we obtain

$$\frac{\frac{\partial f}{\partial a_t^\beta}}{\frac{\partial f}{\partial a_t^0}} = 1 + \frac{D_{t+1}^\alpha}{D_{t+1}^0}. \tag{62}$$

Consider the second part of the right-hand-side of equation (62). For $t = T$, that second part equals 1. For $t = T - 1$, that second part equals something greater than 1, by Lemma 2. Hence, the left-hand-side of equation (62) is greater for $t = T-1$ than for $t = T$, and we have shown that $(a_{T-1}^0 - a_{T-1}^\beta) > (a_T^0 - a_T^\beta)$. We could go through a parallel analysis to show the analogous result for a^β. It follows from equation (59) that inequality (58) is true and the proposition is true for $t = T$, and so $D_{T-1}^\alpha > D_{T-1}^0$. This would be enough to prove Proposition 3 for $t = T - 2$.

It remains to show (58) is true more generally.
Q.E.D.

Proposition 3: *In all but the last period of a t-period game, each player exerts greater effort when even with the other player than if one of them is ahead. If one of them is ahead, the leader exerts greater effort than the follower. For every $t < T$,*

(43) $a_t^\alpha < a_t^0$

and

$$a_t^\beta < a_t^\alpha. \quad (44)$$

Proof: The proof works by recursion, and has four parts.

Part II: Inequality (43).
To prove inequality (43), note that first order conditions (51) and (53) imply that

$$-\frac{\partial g}{\partial a_t^\alpha} D_{t+1}^\alpha = \frac{\partial f}{\partial a_t^0} D_{t+1}^\alpha - \frac{\partial g}{\partial a_t^0} D_{t+1}^0. \tag{63}$$

Since in equilibrium $a_t^0 = b_t^0$ by symmetry, assumption (4) implies that $\frac{\partial f}{\partial a_t^0} = -\frac{\partial g}{\partial a_t^0}$. Hence (63) can be rewritten as

$$\begin{aligned} -\tfrac{\partial g}{\partial a_t^\alpha} D_{t+1}^\alpha &= -D_{t+1}^\alpha + D_{t+1}^0 \tfrac{\partial g}{\partial a_t^0} \\ &= -D_{t+1}^\alpha \tfrac{\partial g}{\partial a_t^0}. \end{aligned} \tag{64}$$

which implies that

$$-\frac{\partial g}{\partial a_t^\alpha} > -\frac{\partial g}{\partial a_t^0}, \tag{65}$$

which implies, given assumption (3), that $a_t^0 > a_t^\alpha$, for any t.

Part III: Inequality (44) for t = T-2.
Proposition 2 tells us that Proposition 3 holds for $t = T-1$, since in a perfect equilibrium the subgame consisting of the last two periods of a T-period game has the same equilibrium as a two-period game. What must be shown is that Proposition 3 also holds for earlier periods. Let us start with $t = T-2$.

To prove inequality (44) for $t = T-2$, note that first order conditions (51) and (52) imply that

$$-\frac{\partial g}{\partial a_t^\alpha} D_{t+1}^\alpha = \frac{\partial f}{\partial a_t^\beta} D_{t+1}^0. \tag{66}$$

Using Lemma 3 to tell us that $D^{\alpha}_{T-1} > D^{0}_{T-1}$, one may conclude from equation (66) that for $t = T - 2$

$$-\frac{\partial g}{\partial a^{\alpha}_{t}} < \frac{\partial f}{\partial a^{\beta}_{t}}. \tag{67}$$

By assumption (5), $-\frac{\partial g}{\partial a^{\alpha}_{t}} = -\frac{\partial f}{\partial b^{\beta}_{t}}$, so equation (67) implies

$$-\frac{\partial f}{\partial b^{\beta}_{t}} < \frac{\partial f}{\partial a^{\beta}_{t}}. \tag{68}$$

By assumption (4), if $a^{\beta}_{t} = b^{\beta}_{t}$ then expression (68) would be an equality. Since it is an inequality, the concavity of f in a combined with assumption (5) tells us that $a^{\beta}_{t} < b^{\beta}_{t}$, and since in equilibrium $b^{\beta}_{t} = a^{\alpha}_{t}$, (44) is proved for the special case of $t = T - 2$.

Part IV: Inequality (44) for General t.

What was special about $t = T - 2$ in Part III was the use of Lemma 3. One may prove inequality (44) for $t = T - 3$ after first proving a version of Lemma 3 for V^{α}_{T-2} instead of V^{α}_{T-1}. Let us use our new knowledge that inequality (44) holds for $t = T - 2$. A generalized version of Lemma 3 is

$$D^{\alpha}_{t} > D^{0}_{t}. \tag{69}$$

The proof of Lemma 3 did not rely on many special characteristics of the last two periods. The numbers "T-1" and "T" can everywhere be replaced by "t" and "t+1", and the only difficulty is that the proof did cite Proposition 2 and equation (23), which are special to periods T and $T - 1$.

Proposition 2 was used to confirm that X_2 and X_3 were positive, which in the T-period game is equivalent to showing that $a^{0}_{T-1} - a^{\alpha}_{T-1} > 0$ and $a^{0}_{T-1} - a^{\beta}_{T-1} > 0$. It was already shown that equation (44) holds for $t = T - 2$, so that can be used to prove that (69) holds for $t = T - 2$. The general form of (23) is

$$D^{\alpha}_{t+1} > VDt + 1^{0}. \tag{70}$$

Lemma 3 itself replaces (23) for the case of $t = T - 2$. Hence, inequality (69) can be proved for $t = T - 2$ using the proof of Lemma 3 with two differences: instead of citing Lemma 1, cite the part of Proposition 3 that was already proved; and instead of citing (23), cite Lemma 3.

Once it is shown that (69) holds for $t = T-2$, one may go back and follow the outline of Part III of this proof to prove inequality (44) for $t = T - 3$. After that is done, one can prove that (69) holds for $t = T - 3$, go back to Part III, and continue recursively, which proves that (44) holds for any value of t.
Q.E.D.

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